38

Revista Brasileira de Física, Vol. 20, nº 1, 1990

Existence of infinite exact eigensolutions for the $x^2 + \lambda x^2/(1+gx^2)$ interaction

Marcus W. Beims and Jason A. C. Gallas
Laboratório de Óptica Quântica, Departamento de Física, Universidade Federal de Santa Catarina, 88049, Florianópolis, Brazil

Received October 30, 1989

Abstract We show that a recent conjecture about the possible existence of an infinite number of exact eigensolution pairs for the $x^2 + \lambda x^2/(1+gx^2)$ interaction having λ and g connected by $\lambda = -(6g^2+4g)$ is true. A constructive method for explicitly obtaining these solutions is given. In addition, we present a REDUCE implementation of the constructive method which allows solution pairs to be easily generated on personal computers.

1. Introduction

In a recent paper[1], we showed that the perturbed harmonic oscillator

$$x^2 + \lambda x^2/(1+gx^2)$$

admits five pairs of exact analytical eigensolutions having the parameters λ and g connected by the relation $\lambda = -6g^2 - 4g$. It was also conjectured that an infinite number of such solution pairs should exist. Our conjecture was proved by Vanden Berghe and Meyer[2] and Lakhtakia[3]. The purpose of this brief paper is to provide a simple constructive proof of the same conjecture. Our proof was obtained by us simultaneously and independently of the aforementioned authors. We believe our result to be of interest because it provides a trivial mean of generating, in principle, all solution pairs. The generation of an arbitrary number of solution pairs can be easily implemented on personal computers able to perform algebra and an example of one such implementation, written in REDUCE, is given here.

The potential $x^2+\lambda x^2/(1+gx^2)$ is of interest in laser physics (as the reduction of the Fokker-Planck equation of a single-mode laser under suitable conditions), in

elementary particle physics (as a one-dimensional Schrödinger equation associated with a zero-dimensional field theory) and in nuclear physics (as being able to reproduce sequences of energy levels in the shell model). (For specific references see, for example, reference 1 and references therein.)

An interesting aspect of the $x^2+\lambda x^2/(1+gx^2)$ interaction with negative λ (as is the case here since $\lambda = -6g^2 - 4g$, $g > 0$) is that the potential behaves asymptotically like a harmonic oscillator but contains a double minimum. Double minimum potentials have been used in the quantum theory of molecules as simple dynamical models to describe the motion of a particle subject to two centers of force. Double minimum potentials are also of great interest in the investigation of diffusive processes in general (quantum tunneling), models for bistable dynamics[4] and in the quantum theory of instantons[5]. The most used examples of double minimum potentials involve functions containing discontinuous derivatives. The potential considered in this paper, as well as its derivatives, is continuous and has a pair of analytical eigensolutions. For a discussion of these matters we refer to the recent review paper of Razavy and Pimpale[6].

The problem we want to address consists of obtaining pairs of simultaneous eigensolutions of the Schrödinger equation

$$\psi'' + [\varepsilon - x^2 - \lambda x^2/(1+gx^2)]\psi = 0, \quad g > 0, \tag{1}$$

for x in the interval $(-\infty, \infty)$, having the generic form

$$\psi_o(x) = \exp\left(-\tfrac{1}{2}x^2\right)(1+gx^2)x, \tag{2}$$

$$\psi_e(x) = \exp\left(-\tfrac{1}{2}x^2\right)(1+gx^2)\varphi_N(x), \tag{3}$$

with

$$\varphi_N(x) = \sum_{i=0}^{N} c_i x^{2i}, \tag{4}$$

where the subindices o and e refer to the symmetry of the eigensolutions. The first five twin solutions have been obtained in reference 1. We now show how to generate twin solutions for arbitrary N.

Substituting ψ_o from equation (2) in the Schrödinger equation (1) one obtains equations yielding

$$\varepsilon_o = 3 - 6g, \tag{5}$$

$$\lambda = -6g^2 - 4g. \tag{6}$$

From the substitution of ψ_e in equation (1) one readily obtains

$$\sum_{i=0}^{N} 2i(2i-1)c_i x^{2i} + \sum_{i=0}^{N} \left(\varepsilon_e + 4gi^2 + 6gi + 2g - 4i - 1\right) c_i x^{2i+2} + \sum_{i=0}^{N} \left(g\varepsilon_e - \lambda - 4gi - 5g\right) c_i x^{2i+4} = 0. \tag{7}$$

From the coefficients of x^{2i} it follows that

$$2i(2i-1)c_i + [2i(2i-1)g - 4i + 3 + \varepsilon_e]c_{i-1} + [3g + g\varepsilon_e - \lambda - 4gi]c_{i-2} = 0, \tag{8}$$

valid for $i = 1, 2, \ldots, N+2$. When $i = N+2$ equation (8) gives

$$4Ng + 5g - g\varepsilon_e + \lambda = 0. \tag{9}$$

Since according to equation (6) we have $\lambda = -6g^2 - 4g$, it follows that

$$\varepsilon_e = 4N - 6g + 1. \tag{10}$$

It is interesting to observe that the energy difference between the two states depends only on N:

$$\Delta\varepsilon = \varepsilon_e - \varepsilon_o = 4N - 2. \tag{11}$$

Equations (6) and (10) may now be used to simplify relation (8), giving the relation

$$c_i = \frac{-1}{2i(2i-1)}\{[2i(2i-1)g - 6g + 4(N-i+1)]c_{i-1} + 4g(N-i+2)c_{i-2}\}, \tag{12}$$

valid for $i = 1, 2, \ldots, N+2$. For $i = 1$ equation (12) gives

$$c_1 = (2g - 2N)c_0. \tag{13}$$

Now, equations (12) and (13) can be used to generate all coefficients c_i appearing in (4) as functions of c_0. For convenience we may set $c_0 \equiv 1$, since the exact normalization is not important here.

The condition that all c_i should vanish for $i > N$ allows us to obtain (from equation (12) with $i = N+1$) a relation between c_N and c_{N-1} :

$$c_N = \frac{-2}{(N+2)(2N-1)} c_{N-1}. \tag{14}$$

The polynomial equation defining the possible g values[1] may now be easily obtained by forcing c_N and c_{N-1} obtained from the recurrence relation (12) to obey the constraint relation (14).

The above results were used to write the following REDUCE program:

```
OFF ECHO $
OPERATOR C $
N      := 5 $
C(0)   := 1 $
C(1)   := 2*(G-N) $
CTE    := -2/((N+2)*(2*N-1)) $
LAST   := CTE*C(N-1) $
FOR I  := 2:N DO
C(I)   := - ( (2*I*(2*I-1)*G - 6*G + 4*(N-I+1))*C(I-1)
              +4*G*(N-I+2)*C(I-2) )/(2*I*(2*I-1)) $
FOR I := 1:N-1 DO WRITE "C(",I,")= ",C(I) $
WRITE "C(",N,")= ",CTE," * C(",N-1,")" $
WRITE " G POLYNOMIAL : " $
WRITE NUM(C(N)-LAST)," = 0 " $
END $
```

This program was implemented on a personal computer and, by changing the value of N on the third program line, used to generate all five solutions presented earlier[1]. The program was further used to generate new twin solutions. Table I presents a summary of the first 15 solutions, together with the corresponding values of ε_o, ε_e and $V_{\min}$, the value of the minimum of the potential. Defining

$R^2 = 2g(3g+2) = -\lambda$ it is easy to see that the minima are located at $x^2_{\min} = (R-1)/g$ and that

$$V_{\min} = -\frac{1}{g}(R-1)^2. \tag{15}$$

Table I - Values of g for which twin solutions exist. $\Delta g \equiv g_N - g_{N-1}$. $\lambda, V_{\min}, \varepsilon_o$ and ε_e are defined in equations (6), (15), (5) and (10) respectively. Note that for $N=1$ and 2 the energy of the even state lies above the relative maximum $V=0$ at $x=0$.

N	g	Δg	λ	$V_{\min}$	ε_o	ε_e
1	0.66667	—	−5.3	−2.6	−1.0	1.0
2	1.45743	0.79076	−18.6	−7.5	−5.7	0.3
3	2.23486	0.77743	−38.9	−12.3	−10.4	−0.4
4	3.00979	0.77494	−66.4	−17.0	−15.1	−1.1
5	3.78383	0.77403	−101.0	−21.7	−19.7	−1.7
6	4.55743	0.77361	−142.9	−26.3	−24.3	−2.3
7	5.33080	0.77337	−191.8	−31.0	−29.0	−3.0
8	6.10403	0.77322	−248.0	−35.6	−33.6	−3.6
9	6.87716	0.77313	−311.3	−40.3	−38.3	−4.3
10	7.65022	0.77306	−381.8	−44.9	−42.9	−4.9
11	8.42323	0.77302	−459.4	−49.6	−47.5	−5.5
12	9.19622	0.77298	−544.2	−54.2	−52.2	−6.2
13	9.96917	0.77295	−636.2	−58.9	−56.8	−6.8
14	10.74210	0.77293	−735.3	−63.5	−61.5	−7.5
15	11.51502	0.77292	−841.6	−68.1	−66.1	−8.1

Figure 1 shows the potential $x^2 + \lambda x^2/(1+gx^2)$ together with two asymptotic potentials (shown as dashed lines): x^2 and $x^2 - 6g - 4$. Superimposed to these potentials we show the solutions ψ_o and ψ_e. The vertical positions of ψ_o and ψ_e correspond to the exact locations of their corresponding eigenenergies. To improve the readability of the figure, the normalization of the eigenfunctions was conveniently chosen so that their total amplitude corresponds to 25% of the full

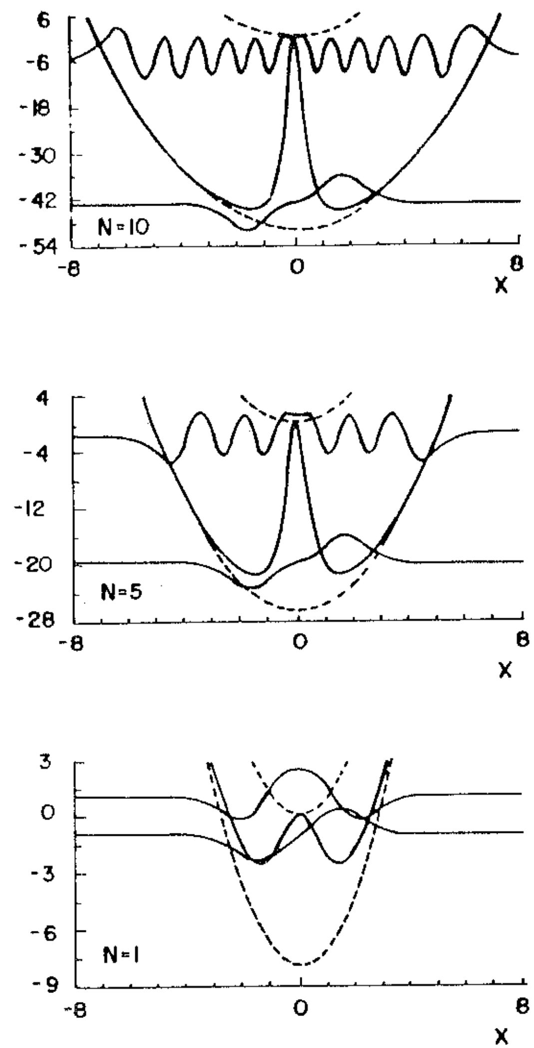

Fig. 1 - Eigenfunctions ψ_o and ψ_e together with the potential $x^2 + \lambda x^2/(1+gx^2)$ (solid curve) and two asymptotic potentials (shown as dashed lines): x^2 and $x^2 - 6g - 4$. The height of the functions corresponds to the exact position of the energy eigenvalues.

scale. All ψ_e were obtained by evaluating numerically on a personal computer the coefficients c_i in eq.(12). As a caveat to the reader we remark that the recurrence relation (12) is very sensitive to the value of g. Preliminary runs using the 6-digit values of g given in Table I, failed to generate the correct eigenfunctions. In particular, the g value used in Figure 1 to generate ψ_e for $N = 10$ was 7.650218591050418350. Sensitivity on parameters is a well-known property of recurrence relations. At this stage we do not see any need for a more stable (possibly backwards) recurrence relation.

In summary, the potential $x^2 + \lambda x^2/(1 + gx^2)$ with λ and g connected by $\lambda = -6g^2 - 4g, \ \ g > 0$ contains an infinite number of closed form eigensolution pairs. The pairs consist of an odd and an even solution having an energy difference depending only on the degree of excitation of the even function (see eq.(11) above). We presented a computer program written in REDUCE allowing the easy generation of these eigensolutions on personal computers. The potential investigated is a quite rare example of a continuous double-minimum potential containing a pair of exact analytical eigensolutions.

MWB thanks the Brazilian agency CAPES for a predoctoral fellowship.
JACG is a research fellow of the Brazilian Research Council(CNPQ).

References

1. J.A.C. Gallas, J. Phys. A: Math. Gen. **21**, 3393 (1988).
2. G. Vanden Berghe and H.E. De Meyer, J Phys A: Math. Gen **22**, 1705 (1989).
3. A. Lakhtakia, J Phys A: Math. Gen **22**, 1701 (1989).
4. N.G. van Kampen, J. Stat. Phys. **17**, 71 (1977).
5. A.M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
6. M. Razavy and A. Pimpale, Phys. Rep. **168**, 305 (1988).

Resumo

Mostramos que uma conjectura recente sobre a possível existência de um número infinito de pares de soluções próprias exatas para a interação $x^2 + \lambda x^2/(1+gx^2)$ com λ e g relacionados por $\lambda = -(6g^2 + 4g)$ é verdadeira. Damos ainda um método construtivo para obter explicitamente estas soluções. Além disso, apresentamos também uma implementação em REDUCE deste método construtivo que permite gerar algebricamente pares de soluções em microcomputadores do tipo PC.