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Accumulation horizons and period adding in optically injected semiconductor lasers

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We study the hierarchical structuring of islands of stable periodic oscillations inside chaotic regions in phase diagrams of single-mode semiconductor lasers with optical injection. Phase diagrams display remarkable *accumulation horizons*: boundaries formed by the accumulation of infinite cascades of self-similar islands of periodic solutions of ever-increasing period. Each cascade follows a specific period-adding route. The riddling of chaotic laser phases by such networks of periodic solutions may compromise applications operating with chaotic signals such as, e.g., secure communications.

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Semiconductor lasers are key components for progress in many areas [1,2], e.g., in the investigations of signatures of wideband chaos synchronization [1,3], for secure communication [2,4], and for ultrahigh-density optical memories [5]. Such compact lasers represent 99.8% of the world market for lasers in terms of units sold per year [6], a market of 600 million units sold in 2003–2004. An important quality of semiconductor lasers is their rich nonlinear response when subjected either to optical injection, to optical feedback, or to modulations. Lasers with optical injection have attracted much attention in recent years, experimentally as well as from theoretical and numerical points of view. The complex phenomenology and intricate structure of bifurcations as a function of the injected intensity and frequency detuning were summarized in a recent survey [7].

The present literature indicates a good overall agreement between theory and experiments [8]. For instance, calculations and numerical simulations predict intricate laser behaviors, including stable periodic oscillations inside regions characterized by chaotic laser signals [9,10]. More recently, Fordell and Lindberg [11] and Chlouverakis and Adams [12] reported diagrams obtained by numerical integration of the rate equations for an optically injected semiconductor laser showing some islands of periodic laser signals embedded in a sea of chaos. These important findings raise an interesting question concerning the precise structuring of laser chaotic phases. In fact, this question is the tip of a much wider problem that we consider here.

Phase diagrams for discrete-time models described by mappings are common nowadays [13,14]. But the much more difficult problem of building detailed phase diagrams for models ruled by sets of nonlinear differential equations has been much less investigated. Of course, diagrams recording complex bifurcations and providing valuable insight for a few of the lowest periods have been obtained in a number of in-depth bifurcation studies using powerful continuation methods [7,15–17]. However, complete diagrams, discriminating simultaneously regions of arbitrarily high periods and regions with chaotic phases, remain essentially unexplored for continuous-time autonomous models. This is the problem we attack here.

Our numerical simulations revealed surprising regularities existing inside the chaotic phases of the laser. As illustrated in Figs. 1 and 2, the parameter space has wide regions characterized by chaotic solutions. These chaotic phases contain both single accumulations as well as accumulations of accumulations. More specifically, chaotic laser phases are riddled with infinite sequences of period-adding cascades, each one converging toward curves that look simple (structureless), denoted "accumulation horizons," for simplicity. One example is indicated by the arrow marked A in Fig. 2(a). From a theoretical point of view, we note that here the differential

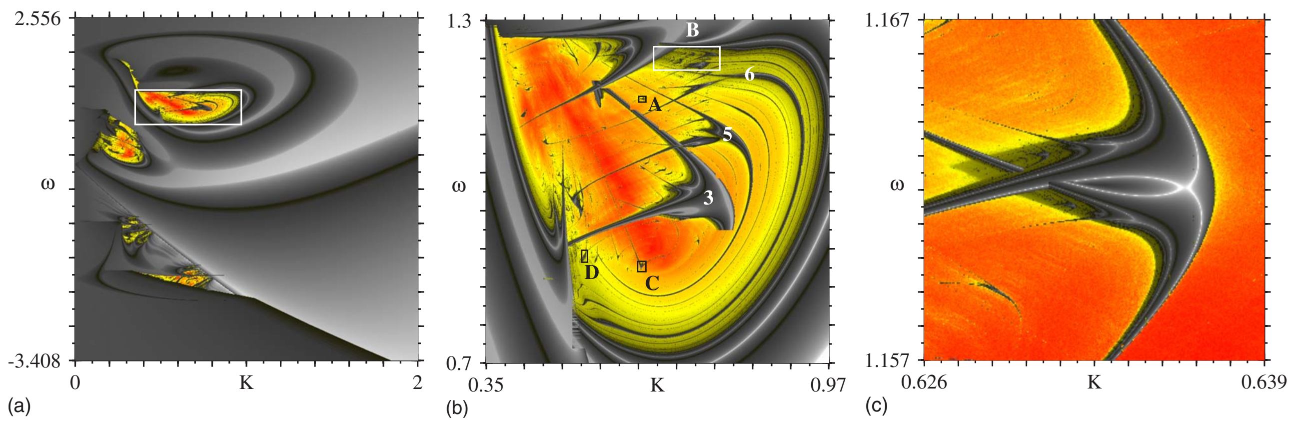

FIG. 1. (Color online) Phase diagrams quantifying both regularity (darker shadings) and chaos (colors; lighter shadings). (a) Global view. (b) Magnification of box in (a), for positive detuning: Numbers denote quantity of peaks in a period of the laser intensity. Boxes A, B, C, and D are shown magnified in the next figures. (c) Magnification of the period-9 island inside box A in (b) showing a structure also found in CO_2 lasers (Ref. [22]). Red denotes "stronger" chaos (more positive Lyapunov exponents).

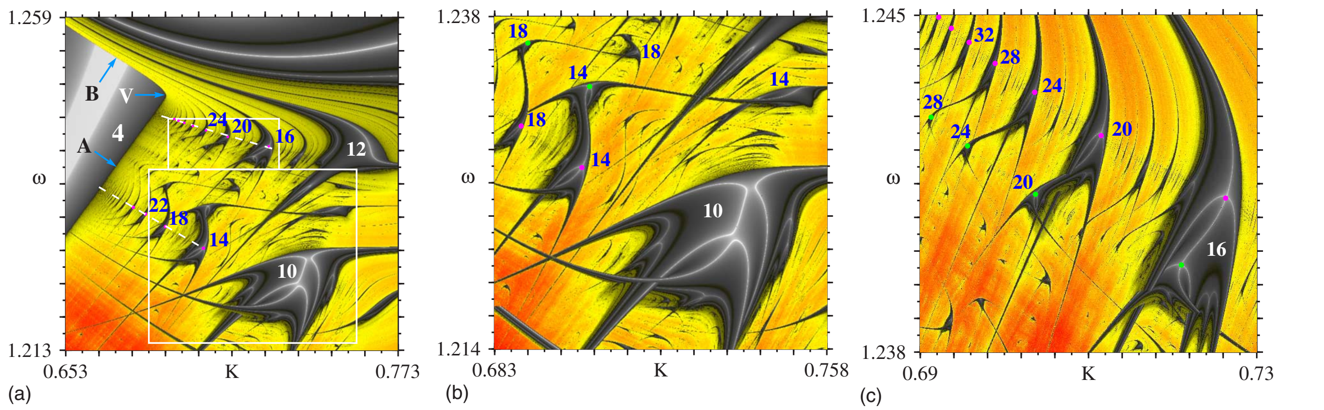

FIG. 2. (Color online) Accumulations of islands of periodic solutions (darker shadings) embedded inside a chaotic phase (yellow-red lighter shadings). (a) Series of islands converging toward a line segment, marked A, forming an *accumulation horizon*. "Legs of periodicity" accumulate parallel to line B. Curves A and B meet at vertex V. Bifurcation diagrams along dotted lines, shown in Fig. 3, display period-adding cascades converging to horizon A, the boundary of a four-peak domain, as indicated. (b) Genesis and separation of distinct $10 \rightarrow 14 \rightarrow 18 \rightarrow \cdots$ period-adding cascades. (c) Similar genesis and separation as in (b) but for distinct $(12) \rightarrow 16 \rightarrow 20 \rightarrow 24 \rightarrow \cdots$ cascades. Numbers refer to the quantity of peaks in the laser intensity.

equations ruling the laser are autonomous equations, i.e., they do not involve time explicitly. Thus, the remarkable organization of the parameter space reported here must originate from an intrinsic interplay between variables and parameters of the laser. We found accumulation horizons to exist abundantly also in electronic circuits, atmospheric and chemical oscillators, and several other systems [18]. To fix ideas, here we focus just on the laser case. Incidentally, we mention that accumulation cascades in optically injected lasers have been investigated by Krauskopf and Wieczorek [17] quite recently. However, their accumulations are of a very different nature than ours [19].

The laser we consider is a single-mode semiconductor laser subjected to monochromatic optical injection, governed by the standard rate equations for the complex laser field $E = E_x + iE_y$ and a population inversion n rescaled such that [8]

$$\dot{E} = K + \left[\frac{1}{2}(1 + i\alpha)n - i\omega\right]E, \tag{1a}$$

$$\dot{n} = -2\Gamma n - (1 + 2Bn)(|E|^2 - 1). \tag{1b}$$

Here, the interesting control parameters are K, the intensity of the injected field, and ω, the detuning frequency. As usual [8], we fix $B=0.0295$, $\Gamma=0.0973$, and $\alpha=2.6$.

Figure 1 illustrates typical high-resolution phase diagrams obtained by computing the spectra of Lyapunov exponents on a 900×900 grid of equally spaced parameters for Eqs. (1a) and (1b), integrated with a standard fourth-order Runge-Kutta scheme with a fixed step size $h=0.01$. Each grid point color-codifies the magnitude of the largest nonzero exponent: negative exponents (indicating periodic solutions) were colored with gray shadings (black indicates zero, white the most negative values), while positive exponents (marking chaotic laser signals) are indicated in a continuously changing yellow-red scale (lighter shadings in black and white). The color scale of individual phase diagrams was renormalized to span each diagram. Figure 1(a) displays the same parameter region investigated recently by Wieczorek *et al.* [8]. To convert ω into GHz, multiply it by 4.6948. Our figure corroborates the low-period bifurcation boundaries reported recently [8] and, more importantly, shows additional details and regularities, such as, e.g., the inner structuring of periodicity domains, the regions where recurring self-similar organizations occur and where they fail to exist. Our figures reveal details that are very hard (if not impossible) to obtain using continuation methods.

Islands of regular laser oscillations in semiconductor lasers were measured by Eriksson and Lindberg in recent experiments [20,21]. First, they were able to identify a period-3 island by tuning the injection intensity for three fixed values of the frequency detuning [20]. Then, by repeating measurements for finer detuning intervals, they cleverly managed to characterize a few islands of low period [21]. Figure 1(b) corroborates such low-periodic islands and shows a myriad of additional islands of ever-increasing periods as discussed below. It also displays several other features, in particular the existence of self-similarities of various kinds. Figure 1(c) displays an island with the familiar shrimp shape [13] recorded when varying two parameters simultaneously (codimension-2 phenomenon). Although well known in discrete-time dynamical systems, this peculiar shape was observed only recently in a nonautonomous continuous-time system, namely, in CO_2 lasers [22].

A series of unexpected and striking accumulation networks may be easily recognized from Fig. 2, presenting successively magnified views of box B in Fig. 1(b). Embedded in the chaotic region, there are regular and abundant networks of stable islands of periodic laser signals with unbounded periodicities. As Fig. 2(a) shows, the parameter networks existing in the chaotic region bridge periodic laser signals of increasingly higher periodicities, which converge systematically toward well-defined and characteristic accumulation boundaries or horizons. As indicated schematically by the numbers in Fig. 2(a), when moving along the dark central bodies of the islands one observes series of period-

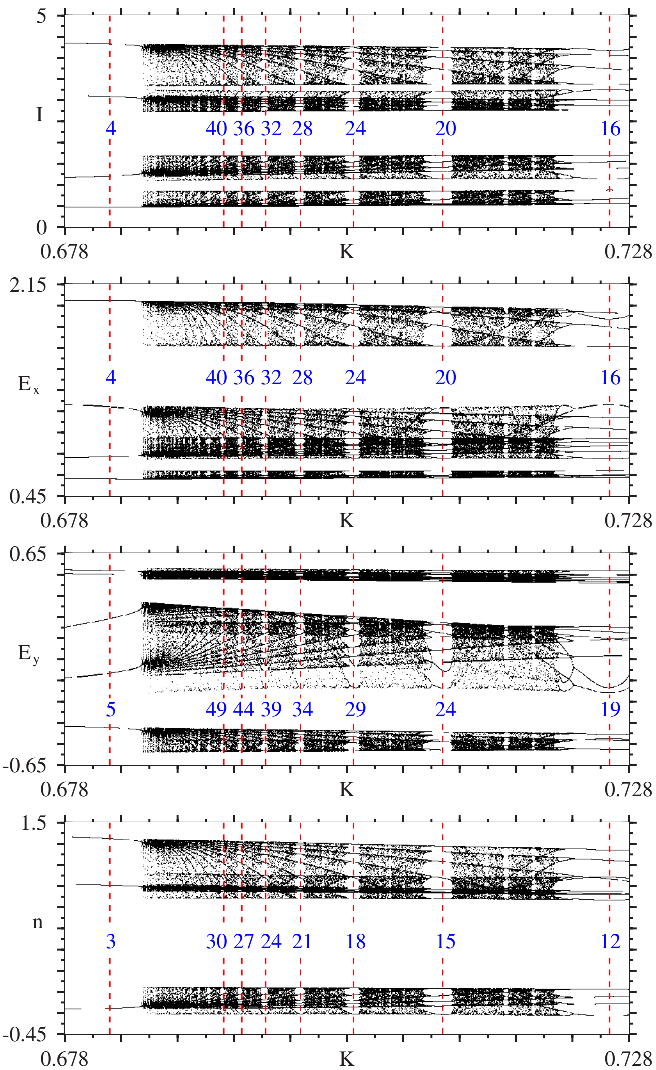

FIG. 3. (Color online) Bifurcation diagrams showing that the number of peaks of the signals depends on the physical quantity being considered. The number of peaks of laser intensity $I \equiv E_x^2 + E_y^2$ coincides with that in E_x.

adding cascades of bifurcations, a characteristic signature of the experimentally elusive and rather challenging homoclinic route to chaos [23–30]. Note that the periodicity in these cascades increases by 4, the same periodicity characterizing the region of periodicity that exists to the left of the accumulation boundary.

That periodicity organizes along specific directions in parameter space is a well-known fact for discrete-time dynamical systems [13]. But that this is also true for continuous-time dynamical system is made obvious now by Fig. 2. A feature not yet reported for discrete-time systems is the original way in which individual period-adding bifurcation cascades are created [18]. As shown by Fig. 2(b), the single period-10 structure (containing the pair of quasiosculating white spines) splits into two essentially separated shrimplike structures [13] as the period increases. This mechanism leads to separate cascades that quickly give the impression of being totally uncorrelated because of the very strong compression experienced by the islands as the period increases more and more without bound. Here, white spines mark loci of the most negative Lyapunov exponents, being loosely equivalent to the superstable loci familiar from discrete-time dynamical systems. The splitting process involves several specific metric properties, for instance, the parameter separation of the islands accumulates to specific values while their volume decreases regularly with characteristic exponents.

The bifurcation diagrams in Fig. 3, obtained when moving along the upper dotted path in Fig. 2(a), show period-adding cascades with the characteristic alternation of chaos and periodicity [23–30]. Numbers labeling periodic windows refer to the number of peaks present in one period of the respective variable. Note the striking fact that different variables display different number of peaks. Since the number of peaks is usually taken to label the "period" of oscillation, one sees that such labels are not unique but depend on the variable used to count the peaks. Note that, independently of the variable selected, the number of periods increases by an amount equal to the number of peaks characterizing the leftmost window, toward which the period-adding cascades accumulate.

As a last noteworthy result found in semiconductor lasers, Fig. 4 illustrates islands of regular signals having the same exquisite shapes found very recently in a rather different sce-

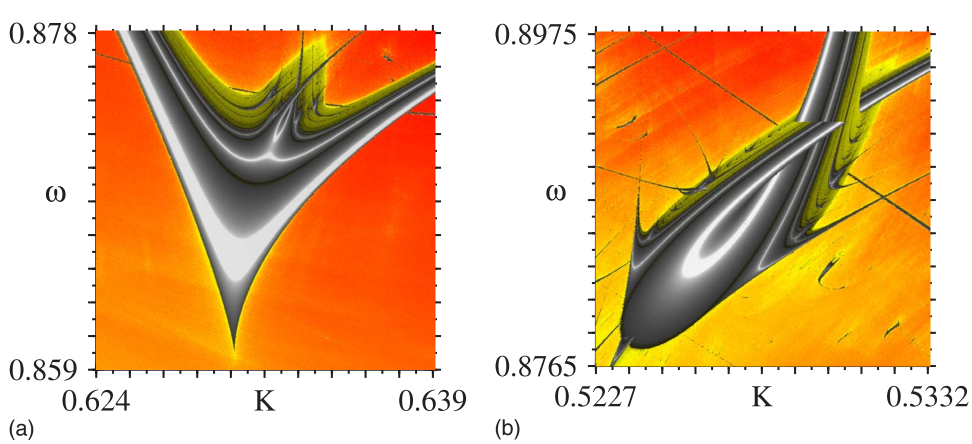

FIG. 4. (Color online) Magnification of boxes C and D in Fig. 1(b), showing typical islands or stable periodic orbits with the same shapes found recently in a completely different scenario: systems without critical points (see text). Color coding as in Figs. 1 and 2.

nario: in a discrete-time dynamical system with no critical points, i.e., in a system not obeying the Cauchy-Riemann conditions [31,32]. Such striking shapes exist abundantly in the lower portion of Fig. 1(b). Thus, semiconductor lasers open the way to investigate experimentally novel and sophisticated mathematical behaviors arising from holomorphic dynamics not ruled by critical points, so far believed to be the key players in the dynamics of complex functions [31].

In summary, chaotic phases of optically injected semiconductor lasers contain peculiar accumulation boundaries and networks formed by stable periodic solutions. Since extended domains of "clean" chaos are critical for a number of laser applications [1,2], these regularities need to be duly taken into account in applications that depend on the existence of wide regions of smooth and continuous chaos, such as secure communication with chaos. Although we concentrated on the case $\alpha=2.6$, representative of the relatively low values more frequently addressed in the literature, larger islands exist for higher values of α, say $\alpha\simeq 6$, making them easier to observe experimentally. Accumulation horizons exist also in other laser systems, e.g., in CO_2 lasers with feedback, and in other sets of differential equations [18].

The accumulation networks reported here pose an interesting question: In sharp contrast with discrete dynamical system, where periodicity varies discretely ("quantized"), an appealing new possibility afforded by lasers is to study how periodicities defined by continuous real numbers evolve and organize in phase diagrams when several parameters are tuned simultaneously. Such investigations should not be too difficult to perform numerically. As a last remark, we briefly mention that the alternating period-chaotic sequences familiar from period-adding cascades [23–30] are in fact an illusory artifact of considering too restricted slices cutting very regular structures in parameter space [18]. The proper unfolding of this phenomenon requires tuning at least two parameters, i.e., is a phenomenon visible only in codimension 2 or higher.

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