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What is the effective impact of the explosive orbital growth in discrete-time one-dimensional polynomial dynamical systems?

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HIGHLIGHTS

- We quantify exactly periodic orbit growth in arbitrary degree polynomial maps.
- We report detailed stability phase diagrams for two cubic and one quartic map.
- Mobius inversion is immensely simpler than kneading sequences.
- Number of phases does not depend on the nonlinearity of the equations of motion.

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ABSTRACT

We study the distribution of periodic orbits in one-dimensional two-parameter maps. Specifically, we report an exact expression to quantify the growth of the number of periodic orbits for discrete-time dynamical systems governed by polynomial equations of motion of arbitrary degree. In addition, we compute high-resolution phase diagrams for quartic and for both normal forms of cubic dynamics and show that their stability phases emerge all distributed in a similar way, preserving a characteristic invariant ordering. Such coincidences are remarkable since our exact expression shows the total number of orbits of these systems to differ dramatically by more than several millions, even for quite low periods. All this seems to indicate that, surprisingly, the total number and the distribution of *stable phases* is not significantly affected by the specific nature of the nonlinearity present in the equations of motion.

1. Introduction

Among the first problems that need to be solved when studying periodic orbits in dynamical systems is that of counting the orbits and their symmetry classes [1–17]. For systems governed by arbitrary equations of motion, the counting problem

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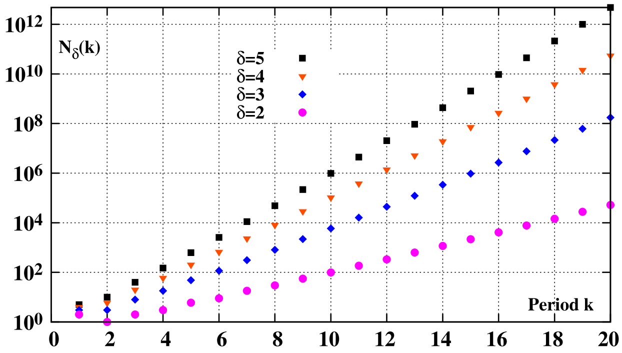

Fig. 1. Growth of the number $N_\delta(k)$ of periodic orbits as a function of the period k for polynomial mappings of degree $\delta = 2, 3, 4, 5$, according to Eq. (5). Already for period as low as 20, cubic dynamics has about 10^3 times more orbits than the quadratic (logistic) map.

has no solution. But for dynamical systems of algebraic origin [18], governed by polynomial equations of motion, it is possible to solve the counting problem exactly, for arbitrary degree of the polynomials, and this is the main analytical result reported here, in Eq. (5). This powerful analytical result harbors surprising practical consequences which are also reported here, in Section 3.

For the celebrated one-dimensional quadratic map, the counting problem was addressed early on, in 1958–63, by Myrberg in a ground-breaking application of computers to dynamics [19–21]. For a survey on one-dimensional quadratic dynamics see Refs. [21,22]. In two-dimensions, the problem was addressed as early as 1979 by Simó [23] for the Hénon map, using an approach centered on the strange attractor creation and destruction. Exact formulas for the number of orbits and orbital symmetries for the $b = -1$ Hamiltonian (area-preserving) limit of the Hénon map $(x, y) \mapsto (a - x^2 + by, x)$ were obtained recently [24]. Knowledge about the proliferation of orbits (Fig. 1) is of relevance for a number of problems, e.g. in arithmetic dynamics, in investigations of *orbital inheritance* dealing with the transformation of orbits into new orbits [17,25], and in the operation of chaotic systems that can perform computations and which are promising candidates for replacing conventional computing technology [26–30].

The direct combinatorial problem of determining, for arbitrary values of control parameters, the partition of the set of orbits either into real and complex orbits or into stable and unstable orbits is very hard. However, the total number of periodic orbits present in polynomial maps of a given degree can be counted in a simple and efficient way. As mentioned, this is the problem solved here. The approach used is a nice application of enumerative combinatorics and the number-theoretic Möbius inversion formula to a key problem in physics and dynamical systems. Several complementary aspects of the combinatorial dynamics of maps are discussed by Alsedà et al. [31].

From a theoretical point of view, our results complement previous studies in this Journal by Xie and Hao [7,8]. They investigated the number of orbits as a function of the number of "laps" of the map, i.e. as a function of the number of monotonic map pieces separated by critical points of the equations of motion. Our results also complement earlier studies concerning the distribution of stability in phase diagrams (control parameter space) of maps [9]. The exact quantification of the orbital distribution is relevant for a number of applications as discussed, for instance, by Lorenz in his last publication [10]. Orbital quantification for maps is also expected to throw light on the distribution of stability in systems governed by differential equations, an exceptionally difficult problem related with Hilbert's sixteenth problem, the still unsolved problem of enumerating limit cycles for polynomial differential equations in the plane [32].

2. Orbital growth in polynomial maps of arbitrary degree

The dynamic state of one-dimensional discrete evolution is governed by the equation of motion

$$x_{t+1} = f(x_t), \tag{1}$$

where the function $f(x)$ defines all specific details connected with the dynamics. For instance, the well-known *logistic map* is defined by $f(x) = \lambda\, x(1 - x)$. Further, Eq. (1) is used nowadays to prospect new directions for computation, where one explores the intrinsic dynamics of a chaotic system for computation [26–30]. More generically, algebraic dynamical systems are governed by polynomial equations of motion of the type

$$f(x) = \sum_{n=0}^{\delta} c_n x^n, \tag{2}$$

where c_n are real control parameters, and $\delta > 1$ is the degree of the mapping $f(x)$.

For a given degree δ, we denote by $N_\delta(k)$ the total number of periodic orbits with a genuine period $k \geq 1$. Denoting by $f^k(x)$ the kth composition of $f(x)$ with itself, the orbital points of a genuine k-periodic orbit are defined by the roots of the polynomial $p_k(x) \equiv x - f^k(x)$. Here, *genuine* is used to stress the fact that all roots of $p_k(x)$ are necessarily distinct, i.e. that

Table 1
Growth of the total number $N_\delta(k)$ of genuine period-k orbits for dynamics governed by one-dimensional algebraic equations of motion of degree δ, illustrated here for $\delta = 2, 3, 4, 5, 6$.

k	$N_2(k)$	$N_3(k)$	$N_4(k)$	$N_5(k)$	$N_6(k)$
1	2	3	4	5	6
2	1	3	6	10	15
3	2	8	20	40	70
4	3	18	60	150	315
5	6	48	204	624	1554
6	9	116	670	2580	7735
7	18	312	2340	11160	39990
8	30	810	8160	48750	209790
9	56	2184	29120	217000	1119720
10	99	5880	104754	976248	6045837
11	186	16104	381300	4438920	32981550
12	335	44220	1397740	20343700	181394535
13	630	122640	5162220	93900240	1004668770
14	1161	341484	19172790	435959820	5597420295
15	2182	956576	71582716	2034504992	31345665106
16	4080	2690010	268431360	9536718750	176319264240
17	7710	7596480	1010580540	44878791360	995685849690
18	14532	21522228	3817733920	211927516500	5642219252460
19	27594	61171656	14467258260	1003867701480	32071565263710
20	52377	174336264	54975528948	4768371093720	182807918979777

by iterating Eq. (1) one obtains k distinct values of x_t, before the sequence starts to repeat. Obviously, the iteration of Eq. (1) may repeat after j steps, for some $j < k$. But in such case the orbital period would not be k, but j, a different (and smaller) period. Since period-1 orbits (i.e. fixed points) repeat indefinitely upon iteration, they clearly qualify as periodic orbits of any arbitrary period length. But such orbits are all trivial orbits that do not qualify as genuine orbits for any period higher than 1. They are genuine orbits of period 1.

For a fixed dynamical system, characterized by a polynomial of degree δ, fixed, the total number of roots of $p_k(x) = x - f^k(x)$ is δ^k. These roots are obviously points of orbits with period k or of orbits with period which are divisors d of k. Thus,

$$\delta^k = \sum_{d|k} dN_\delta(d). \tag{3}$$

Now, Möbius inversion [33] yields

$$k\,N_\delta(k) = \sum_{d|k} \mu(d)\delta^{k/d} \tag{4}$$

so that the number of periodic orbits is then, simply,

$$N_\delta(k) = \frac{1}{k}\sum_{d|k} \mu(d)\delta^{k/d}. \tag{5}$$

In this equation, $\mu(d)$ is the Möbius function [33] and the sum runs over all divisors d of the period k, including the trivial divisors $d = 1$ and $d = k$.

Eq. (5) is our key result. It is valid for arbitrary degrees δ and for arbitrary periods k. For the lowest nonlinearities, namely for $\delta = 2$, 3, 4 and 5, Fig. 1 illustrates how fast the number of orbits grows as a functions of the period k. The numbers plotted in this figure are from Table 1, generated with Eq. (5). As is clear from Fig. 1, even for a period as low as 20, cubic dynamics has about 10^3 times more orbits than the quadratic (logistic) map. Considerably more dramatic ratios are obtained comparing the number of orbits for polynomials of higher degrees. For the particular case of the quadratic map, $\delta = 2$, our values for $N_2(k)$ ought to agree with the corresponding ones for $m = 2$ listed in Table IV of Xie and Hao [7], but they do not. We do not know the origin of the differences. Our values, however, agree with those found by MacKay, as discussed in Ref. [16].

Although Eq. (5) is not particularly hard to obtain and should have been known for a long time, we have not been able to find it in the literature. Knowledge of the exact number of orbits is, however, fundamental in many applications, particularly when classifying analytically all possible behaviors as a function of parameters [17] or when truncating sums over all periodic orbits in computations involving trace formulas [25]. Knowledge of the exact orbital growth leads naturally to interesting questions, e.g. what is the mechanism preventing a much larger number of orbits of becoming stable? Noteworthy is the fact that, for arbitrary periods and dimensions, the number of orbits given by Eq. (5) depends neither on the number of "laps" of the polynomial nor on any associated parametrization in terms of "kneading sequences" [7,8,24]. Möbius inversion suffices.

3. Invariance of the number of stable orbits

Since the maps display an extremely strong proliferation of orbits as a function of the nonlinearity, it is natural to ask about the impact of the several millions of orbits in the control parameter planes of the individual maps. To address this

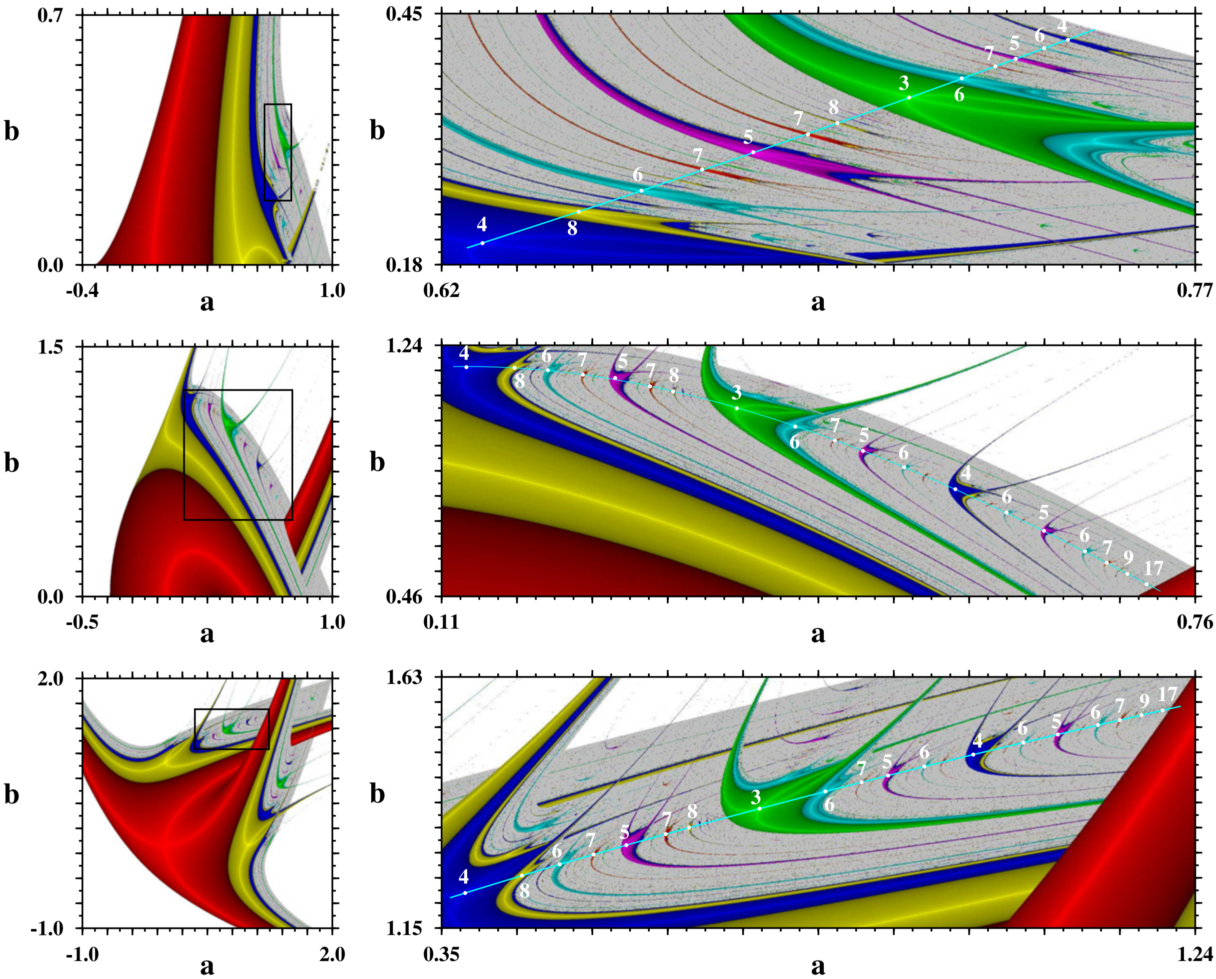

Fig. 2. Phase diagrams for the two normal forms of the cubic map (top rows) and for quartic map (bottom row). Although these phase diagrams look rather similar, Eq. (5) demonstrates that the underlying dynamics may differ by several thousands of orbits. Colors mark periodic orbits, gray shading denotes chaos, and white divergence. The white parabolas intersecting inside each periodicity island represent loci of superstable orbits. White dots indicate the location of doubly superstable orbits and numbers refer to their period. The most easily visible points are located at the central body of infinite symmetrical cascades of the stability domains, part of "shrimps" [9,34]. See text.

question, in Fig. 2 we present phase diagrams computed with high-resolution for extended regions of the parameter space for representative systems, namely for the pair of normal forms which characterize every possible system with cubic dynamics, and for a quartic map. In this figure, the top row shows phase diagrams for $f(x) = x(x^2 - 3a) - b$, the center row for the dual cubic $f(x) = -x(x^2 - 3a) - b$, while the bottom row shows phase diagrams for the standard canonical quartic $f(x) = (x^2 - a)^2 - b$ [9,11–13]. Boxes seen in the panels on the left column are shown magnified on the right panels. All these diagrams display Lyapunov exponents computed and plotted in the usual way [12,13]. Each individual panel displays the analysis of a mesh of 900×900 equally spaced parameter points. Iterations were started from $x = 0$. As usual in these calculations [12,13], the first 10^4 iterations were disregarded as transient behavior. In presence of multistability, initial conditions were chosen in order to maximize visibility of periodic windows. Colors refer to periodic motions while the gray shading denotes chaos (lack of periodicity). Numbers mark the period of the main stability domains and demonstrate the great similarity of the distribution of periodic orbits, despite differences of the underlying maps.

Fig. 2 demonstrates forcefully that all phase diagrams look rather similar, with identical phase ordering, despite their distinct nonlinearity and the strong orbital proliferation recorded in Table 1. So, where do all additional orbits end up? First, periodic orbits exist continuously, independently of the parameter values, although many of them might exist only in the complex sector of the phase space. Second, our formula makes no difference between real, complex, stable or unstable orbits, simply summing them all together. Therefore, the resemblance of the three phase diagrams might be thought as indicative that the orbital proliferation occurs mainly among orbits that are either complex or unstable, or both. Very surprisingly, the number of stable orbits does not seem to be significantly affected by the specific degree of the nonlinearity. Clearly, to clarify what is happening one needs to find a way to compute separately each class of orbits as a function of the parameters. Although rather enticing, this task seems far from trivial.

From the regular ordering of the stable periodic orbits seen in Fig. 2 one might be led to wonder if such ordering could have something in common with Sharkovsky's ordering [35–39] or with its much celebrated corollary "period-three implies chaos" [40]. Although *a priori* the existence of such connection cannot be ruled out, it seems to be rather unlikely. For, the aforementioned theorems deal with existence proofs of essentially *unique unstable orbits* with periods obeying a strict sequential ordering without repetitions, not with what we deal here, namely with *multiple stable orbits* which emerge in a sort of infinitely cascaded organization where periods may repeat abundantly as they grow.

4. Conclusions and outlook

In conclusion, we quantified exactly the growth of periodic orbits in polynomial maps and computed high-resolution phase diagrams for cubic and quartic systems. Our result is valid for arbitrary periods and dimensions, with the number of orbits given by Eq. (5) depending neither on the number of "laps" of the polynomial nor on any associated parametrization in terms of the burdensome kneading sequences [7,8,24]. Möbius inversion suffices. Further, we found the stability phases of the maps considered to emerge all distributed in a similar way, preserving a characteristic invariant ordering among them. These coincidences are remarkable since the number of orbits of these systems differs dramatically by more than several millions, even for quite low periods. All this seems to indicate that the total number and the distribution of *stable phases* is not significantly affected by the specific nature of the nonlinearity present in the equations of motion.

A challenging open problem is to find a means of determining separately the number of real, complex, stable, and unstable orbits that coexist as a function of control parameters and symmetry classes for such orbits. The exact determination of symmetry classes for the Hénon Hamiltonian repeller can be found in Table 1 of Ref. [16]. The problem of counting the periodic orbits of linear maps on a torus is proposed as a challenge in the book of Alligood et al. [41]. Flatto and Lagarias discussed some counting-problems associated with the statistics of orbits on the Lorenz attractor [42]. Estimates of the number of unstable periodic orbits in noisy chaotic systems are given by Pei et al. [43], but no exact and general results are known. As the degree of the polynomial governing the dynamics grows, why the corresponding explosive growth of the number of periodic orbits seems not to significantly affect the number of *stable* orbits? Why does this number seem to remain invariant? What is the effective role of multistability in all this?

Note added: After submitting this paper, we found a result given on page 168 of a book by Rotman [44] showing that, for prime δ, our Eq. (5) for $N_\delta(k)$ coincides with the equation obtained in a rather different context, for the number of irreducible polynomials of degree k over GF(δ). Rotman refers to Simmons [45] for a proof of this result. Our equation, however, is valid for any value of δ, prime or not. It would be interesting to interpret the values obtained for non-prime δ in the context of Galois fields. We are indebted to Sebastian van Strien, Franco Vivaldi, and Patrick Morton for pointing out that formulas to count orbits of polynomial maps are also presented in Refs. [46–50].

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