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Polynomial interpolation as detector of orbital equation equivalence

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Equivalence between algebraic equations of motion may be detected by using a p-adic method, methods using factorization and linear algebra, or by systematic computer search of suitable Tschirnhausen transformations. Here, we show standard *polynomial interpolation* to be a competitive alternative method for detecting orbital equivalences and field isomorphisms. Efficient algorithms for ascertaining equivalences are relevant for significantly minimizing computer searches in theoretical and practical applications.

Keywords: Polynomial equivalence; polynomial isomorphism; algebraic computation.

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1. Introduction

Massive simulations in computer clusters have revealed that the control parameter space of dissipative dynamical systems is riddled with stability islands characterized individually by periodic motions of ever-increasing periods which accumulate, exhibiting conspicuous and interesting regularities. Even in systems governed by *simple* polynomial maps, the number of periodic orbits displays an explosive growth as a function of the nonlinearity.[1] A few years ago, it was realized that the nucleation of stability in classical systems occurs in a variety of ways which normally involve the

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presence of peak-doubling and peak-adding cascades, extending over wide regions of the control space. For instance, transitions among distinct stable oscillatory phases may, or may not, be mediated by parameter intervals, windows, of chaotic oscillations.[2–4] Details of these and other novel regularities are summarized in recent surveys concerning complexities observed in the accumulations of doubling and adding cascades in laser systems (see Ref. 5) in chemistry,[6] in biochemical models,[7,8] and in the dynamics of cancer.[9]

Efficient algorithms for explicitly computing field isomorphisms and their inverses for dynamical systems governed by polynomial maps are central players for analytically assessing the aforementioned regularities present in both theoretical and practical applications. Polynomial maps are useful because metric properties of systems governed by sets of nonlinear differential equations cannot yet be analytically determined due to the lack of methods to solve them exactly. Since stability windows quickly become very narrow as oscillations increase, accuracy is tantamount to exact analytical work. One added advantage of polynomial maps is that the equations of motion generated by them are always exact, not contaminated by the unavoidable round-off and discretization errors, arising from numerical approximations of differential equations.

Currently, the common methods used for determining field isomorphism and equivalence among equations of motion are: (i) a p-adic method reported by Zassenhaus and Liang and used[10,11] to study isomorphisms among quintic polynomials of all three *signatures*[12](n, ℓ), namely $(1, 2)$, $(3, 1)$ and $(5, 0)$, where n refers to the number of real roots, while ℓ refers to the number of pairs of complex roots; (ii) methods using factorization and linear algebra[13]; (iii) a systematic search of suitable Tschirnhausen transformations constrained by some quantity of interest, usually polynomial discriminants. This approach is efficient for systems with low-degree equations of motion.[14]

The purpose of this paper is to introduce an alternative method to detect equivalence among orbital equations of motion, namely Lagrange polynomial interpolation. While standard methods look for isomorphisms, focusing primarily on properties of the number fields involved, the method based on polynomial interpolation seeks isomorphisms directly among irreducible polynomials, because they are the objects that arise automatically as equations of motion governing periodic trajectories of dynamical systems of algebraic origin.[15–17] After obtaining sets of equations of motion, the most important task in dynamics is to determine whether or not such equations are interconnected (i.e. define or not the same number field), and when they are, to obtain complete sets of explicit expressions for the transformations and corresponding inverses interconnecting them.

2. Polynomial Interpolation as Isomorphism Detector

Although already used in 1779 by Edward Waring and an easy consequence of a formula published in 1783 by Euler, "Lagrange" interpolation is a result rediscovered by Lagrange in 1795, which is nowadays traditionally used in numerical analysis for polynomial interpolation. The history of polynomial interpolation is however considerably older than the works above, as reviewed by Meijering.[18]

For a given set of k pairs (a_i, b_i), $i = 1, \ldots, k$ with all a_i distinct, the interpolation polynomial is defined as the lowest degree polynomial that assumes the value b_i at the point a_i for each i. The interpolation polynomial is a linear combination

$$L(x) = \sum_{i=0}^{k} b_i \ell_i(x),$$

where

$$\ell_i(x) = \frac{(x-a_1)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_k)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_k)} = \prod_{j\neq i}\frac{x-a_j}{a_i-a_j}$$

are the basis polynomials. The interpolator $L(x)$ has degree $k' \leq (k-1)$ and $L(a_1) = b_1, \ldots, L(a_k) = b_k$. In all cases of interest here, not only are the a_i distinct but the b_i are too, so we may reverse the interpolations.

Now, it is not difficult to see that by taking $b_1 = a_2, \ldots, b_{k-1} = a_k, b_k = a_1$ we obtain $L(a_1) = a_2, \ldots, L(a_k) = a_1$ or, in other words, for this choice of b_i the "action" of $L(x)$ is to induce a (cyclic) permutation among the elements $a_1, \ldots, a_k$. This is the basic observation that will be explored in the remainder of the paper as an efficient detector of equivalence and isomorphism among polynomial equations of motion. In the applications considered here, the elements a_i and b_i are usually roots of algebraic orbital equations which have real coefficients and, consequently, may be real or complex numbers. Depending on the numerical values of (a_i, b_i), the transformations resulting from permutations of the b_i will have coefficients defined by real or complex numbers. Isomorphisms of interest to us here are the ones characterized by rational coefficients. As seen in the examples in the next section, in most cases, these interesting isomorphisms involve just integer coefficients.

3. Applications

3.1. *Equivalences of Vandermonde's totally real cyclic quintic*

As a first application, we apply polynomial interpolation to find direct and inverse transformations that establish the equivalence among a pair of cyclic quintics of minimum discriminant $\Delta = 14641 = 11^4$ originally considered by Cohn,[14,19] namely

$$V(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1, \quad \text{(Vandermonde's quintic)} \tag{1}$$

$$G(x) = x^5 + 2x^4 - 5x^3 - 2x^2 + 4x - 1. \tag{2}$$

The polynomial $V(x)$ represents a period-five orbital equation of motion for at least three paradigmatic physical models in the so-called *generating partition limit*,[20] namely the quadratic map $x_{t+1} = 2 - x_t^2$, the Hénon map $(x, y) \mapsto (2 - x^2, y)$, and the canonical quartic map,[21,22] namely $x_{t+1} = (x_t^2 - 2)^2 - 2$. For details see, e.g. Refs. 23 and 24. Apart from these fruitful applications, $V(x)$ is the celebrated quintic solved by Vandermonde (1735–1796) using "Lagrange" resolvents a number of years *before* Lagrange, and radical expressions at a time when it was still unknown that a general solution of quintic equations was not possible.[25–27]

To define basis polynomials and interpolations, we fix the roots of $V(x)$ and $G(x)$ in the following orders:

$$v_1 \simeq -1.68, \quad v_2 \simeq -0.83, \quad v_3 \simeq 0.28, \quad v_4 \simeq 1.30, \quad v_5 \simeq 1.91,$$
$$g_1 \simeq -3.22, \quad g_2 \simeq -1.08, \quad g_3 \simeq 0.37, \quad g_4 \simeq 0.54, \quad g_5 \simeq 1.39.$$

Fixing $a_i = v_i$, we compute the five basis elements $\ell_i(x)$. To check for the existence of transformations allowing the passage from $V(x)$ to $G(x)$, one needs to consider the 120 permutations of the roots g_i, using each permuted set of roots as the points b_i. This generates a set $\{L_n(x)\}$ of transformations, for $n = 1, \cdots, 120$. Proceeding in this way, we find that, although the transformations may have both real and complex coefficients, some permutations produce transformations having just *integer* or *rational* coefficients. For the passage from $V(x)$ to $G(x)$, we find five transformations given in the upper part of Table 1, together with the root permutations leading to them. To obtain the inverse transformations, allowing the passage from $G(x)$ to $V(x)$, we fix $a_i = g_i$ and investigate the nature of the coefficients of the 120 transformations obtained by taking $b_i = v_i$ for all possible permutations of the roots v_i. As before, we find five inverse transformations, also listed in Table 1 with the root permutations leading to them. To obtain the transformations, we wrote a program for Maple 2018 (X86 64 LINUX) running on a Dell XPS 13 notebook. Recently, using systematic search of coefficients,[14] it was possible to discover the nine new transformations that had not been found with p-adic methods. Systematic coefficient search of the 10 transformations required about 2.1 s and used 133.1 MB of memory. In contrast, Lagrange interpolation has enabled these transformations to be obtained in just 0.25 s and using only 13.7 MB of memory, a considerable

Table 1. The 10 transformations interconnecting $V(x)$ and $G(x)$.

b_1	b_2	b_3	b_4	b_5	Direct transforms: $V(x) \to G(x)$
g_4	g_1	g_2	g_5	g_3	$D_1 = -x^3 + x^2 + 3x - 2$
g_5	g_2	g_4	g_3	g_1	$D_2 = -x^3 + 2x$
g_1	g_5	g_3	g_2	g_4	$D_3 = x^3 - x^2 - 2x + 1$
g_3	g_4	g_5	g_1	g_2	$D_4 = x^4 - 4x^2 - x + 2$
g_2	g_3	g_1	g_4	g_5	$D_5 = -x^4 + x^3 + 4x^2 - 2x - 3$
b_1	b_2	b_3	b_4	b_5	Inverse transforms: $G(x) \to V(x)$
v_2	v_3	v_5	v_1	v_4	$I_1 = 4x^4 + 10x^3 - 15x^2 - 15x + 9$
v_2	v_3	v_1	v_4	v_5	$I_2 = x^4 + 2x^3 - 5x^2 - 3x + 3$
v_4	v_1	v_2	v_5	v_3	$I_3 = -2x^4 - 5x^3 + 7x^2 + 7x - 3$
v_3	v_4	v_5	v_1	v_2	$I_4 = -x^4 - 2x^3 + 5x^2 + 2x - 3$
v_3	v_1	v_2	v_4	v_5	$I_5 = -2x^4 - 5x^3 + 8x^2 + 9x - 5$

improvement. This speedup of the classification of orbital points is, of course, a desirable feature, that opens the possibility of investigating orbital equations of considerably higher degrees.

While a systematic search of suitable Tschirnhausen transformations was able to find the transformations in Table 1, the interpolation polynomials introduced here have the advantage of revealing concomitantly, as a byproduct, the nature of the action of the individual transformations on the roots. We remark that some of the transformations in Table 1 are reducible, e.g. $D_2(x)$ is clearly reducible, as also are D_1, D_4, I_2, and I_3.

3.2. *Hasse's problem: Equivalence of quintics with complex roots*

As a second application, we use polynomial interpolation to uncover four new transformations, Eqs. (3)–(6), providing a complete and "symmetric solution", i.e. a solution providing both *direct* and *inverse* connections for a classical problem posed by Hasse, who conjectured the possible existence of an isomorphism between three quintics sharing a factor 47^2 in their discriminant.[28,29] Such isomorphism was indeed confirmed by Zassenhaus and Liang,[10] who used a p-adic method to uncover a pair of generating automorphisms of the Hilbert class field over $\mathbb{Q}(\sqrt{-47})$. Their pair of transformations is certainly enough to establish isomorphism of the quintics but, as mentioned, does not provide an unbiasedly balanced and symmetric solution to Hasse's problem, i.e. a solution containing all possible *direct* and *inverse* transformations among all polynomials involved.

Hasse's problem is concerned with relations between the zeros of three quintic equations obtained by Weber,[30] by Fricke,[31] and by Hasse,[28,29] while investigating class invariants for modular equations with discriminant −47. The three quintics found by these authors are, respectively,

$$\begin{aligned} f_W &= x^5 - x^3 - 2x^2 - 2x - 1, && \theta_W \text{ the real root;} \\ f_F &= x^5 - x^4 + x^3 + x^2 - 2x + 1, && \theta_F \text{ the real root;} \\ f_H &= x^5 + 10x^3 - 235x^2 + 2610x - 9353, && \theta_H \text{the real root.} \end{aligned}$$

As reported by Zassenhaus and Liang,[10] Hasse asked whether or not $\theta_W, \theta_F, \theta_H$ generate the same field. And if so, how to express these roots in terms of each other?

Zassenhaus and Liang demonstrated that the polynomials indeed generate the same field as manifest by the following transformations:

$$\theta_H = 5\theta_W^2 - 5\theta_W - 2,$$

$$\theta_W = -\theta_F^4 - 2\theta_F + 1.$$

Proceeding as before, we find the additional root interconnections which read, in the notation used by Zassenhaus and Liang:

$$\theta_F = -\theta_W^4 + \theta_W^3 + \theta_W + 1, \tag{3}$$

$$\theta_H = 10\theta_F^4 - 5\theta_F^3 + 5\theta_F^2 + 10\theta_F - 12, \tag{4}$$

$$\theta_W = \frac{1}{6875}(6\theta_H^4 + 23\theta_H^3 + 194\theta_H^2 - 1308\theta_H + 9821), \tag{5}$$

$$\theta_F = -\frac{1}{6875}(\theta_H^4 + 13\theta_H^3 + 179\theta_H^2 + 717\theta_H - 444). \tag{6}$$

which may be easily verified. Note the conspicuous presence of noninteger coefficients in Eqs. (5) and (6), not a common occurrence in the literature. Thus, polynomial interpolation is also able to deal with situations involving not only real but also complex roots. Of course, not only the real roots but also all the complex roots are properly transformed by the same transformations above. When added to the known pair, the four new transformations reported here solve Hasse's problem completely and symmetrically, with no bias.

As described by Pohst,[32] Hasse's problem played an important role in establishing computational algebraic number theory at a time when computations of all kinds were taboo. In the early 1960s, Zassenhaus developed algorithmic means for number theoretical experiments in algebraic number theory. A major success was the proof of the isomorphism of the three quintic fields which occurred as candidates for the real subfield of the Hilbert class field of $\mathbb{Q}(\sqrt{-47})$. This problem, pointed out by Hasse, could not be solved by theoretical methods. It was the numerical solution by Zassenhaus and Liang[10] that gave major credit to methods in constructive algebraic number theory.

3.3. *Equivalence among totally real sextics with small coefficients*

A very interesting and much studied class of equations of motion involves sextic polynomials.[33] Their splitting fields may contain quadratic and cubics subfields or no subfields at all. Thus, sextics may appear generically as orbital clusters algebraically entangling orbits into distinct groups of periodicity. Of particular interest, is knowledge concerning cyclic sextics. The minimum discriminant of sextics, 300, $125 = 5^3 \cdot 7^4$, was found by Liang and Zassenhaus for the polynomial $s_1(x)$ a totally real cyclic sextic[34]:

$$s_1(x) = x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1, \tag{7}$$

$$s_2(x) = x^6 + x^5 - 7x^4 - 2x^3 + 7x^2 + 2x - 1, \tag{8}$$

$$s_3(x) = x^6 - 2x^5 - 7x^4 + 2x^3 + 7x^2 - x - 1, \tag{9}$$

$$s_4(x) = x^6 + 2x^5 - 7x^4 - 2x^3 + 7x^2 + x - 1. \tag{10}$$

The same minimum discriminant is also shared by $s_2(x), s_3(x), s_4(x)$, simple reincarnations of $s_1(x)$ after the substitutions $x \to \pm x$ and $x \to \pm 1/x$ and suitable simplifications. Curiously, the roots of the above polynomials are interconnected in subtle ways by a multitude of transformations, given in Tables 2 and 3. The values of

Table 2. Direct transformations among the $s_i(x)$ of Eqs. (7)–(10).

b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_1(x) \to s_2(x)$
1	4	2	6	5	3	$2x^5 - x^4 - 14x^3 - 4x^2 + 10x + 2$
2	1	3	4	6	5	$-3x^5 + x^4 + 21x^3 + 9x^2 - 11x - 4$
3	2	5	1	4	6	$-6x^5 + 2x^4 + 43x^3 + 17x^2 - 28x - 7$
4	6	1	5	3	2	$4x^5 - 2x^4 - 28x^3 - 7x^2 + 18x + 2$
5	3	6	2	1	4	$3x^5 - 22x^3 - 15x^2 + 12x + 6$
6	5	4	3	2	1	$-x$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_1(x) \to s_3(x)$
1	2	5	3	6	4	$-7x^5 + 2x^4 + 50x^3 + 22x^2 - 30x - 8$
2	3	1	6	4	5	$6x^5 - 2x^4 - 43x^3 - 17x^2 + 29x + 7$
3	6	2	4	5	1	$-x^4 + x^3 + 6x^2 - x - 2$
4	5	6	1	2	3	$-x^5 + x^4 + 7x^3 - 2x^2 - 7x + 2$
5	1	4	2	3	6	$-2x^5 + x^4 + 14x^3 + 4x^2 - 9x - 2$
6	4	3	5	1	2	$4x^5 - x^4 - 29x^3 - 13x^2 + 18x + 5$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_1(x) \to s_4(x)$
1	3	4	2	6	5	$-4x^5 + x^4 + 29x^3 + 13x^2 - 18x - 5$
2	6	3	5	4	1	$2x^5 - x^4 - 14x^3 - 4x^2 + 9x + 2$
3	2	1	6	5	4	$x^5 - x^4 - 7x^3 + 2x^2 + 7x - 2$
4	1	5	3	2	6	$x^4 - x^3 - 6x^2 + x + 2$
5	4	6	1	3	2	$-6x^5 + 2x^4 + 43x^3 + 17x^2 - 29x - 7$
6	5	2	4	1	3	$7x^5 - 2x^4 - 50x^3 - 22x^2 + 30x + 8$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_2(x) \to s_3(x)$
1	5	4	2	6	3	$-x^4 - x^3 + 6x^2 + x - 2$
2	1	5	3	4	6	$-4x^5 - x^4 + 29x^3 - 13x^2 - 18x + 5$
3	2	1	6	5	4	$x^5 + x^4 - 7x^3 - 2x^2 + 7x + 2$
4	6	3	5	2	1	$7x^5 + 2x^4 - 50x^3 + 22x^2 + 30x - 8$
5	4	6	1	3	2	$-6x^5 - 2x^4 + 43x^3 - 17x^2 - 29x + 7$
6	3	2	4	1	5	$2x^5 + x^4 - 14x^3 + 4x^2 + 9x - 2$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_2(x) \to s_4(x)$
1	4	5	3	6	2	$-2x^5 - x^4 + 14x^3 - 4x^2 - 9x + 2$
2	3	1	6	4	5	$6x^5 + 2x^4 - 43x^3 + 17x^2 + 29x - 7$
3	1	4	2	5	6	$-7x^5 - 2x^4 + 50x^3 - 22x^2 - 30x + 8$
4	5	6	1	2	3	$-x^5 - x^4 + 7x^3 + 2x^2 - 7x - 2$
5	6	2	4	3	1	$4x^5 + x^4 - 29x^3 + 13x^2 + 18x - 5$
6	2	3	5	1	4	$x^4 + x^3 - 6x^2 - x + 2$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_3(x) \to s_4(x)$
1	3	2	5	4	6	$2x^5 - 6x^4 - 7x^3 + 8x^2 + 3x - 1$
2	6	5	1	3	4	$-2x^4 + 6x^3 + 7x^2 - 8x - 2$
3	2	6	4	1	5	$x^4 - 2x^3 - 6x^2 + 2$
4	1	3	6	5	2	$-2x^5 + 7x^4 + 4x^3 - 12x^2 + 3x + 2$
5	4	1	2	6	3	$-x^3 + 3x^2 + 3x - 3$
6	5	4	3	2	1	$-x$

Table 3. Inverse transformations among the $s_i(x)$ of Eqs. (7)–(10).

b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_2(x) \to s_1(x)$
1	3	6	2	5	4	$-6x^5 - 2x^4 + 43x^3 - 17x^2 - 2x + 7$
2	1	3	4	6	5	$-3x^5 - x^4 + 21x^3 - 9x^2 - 11x + 4$
3	6	5	1	4	2	$3x^5 - 22x^3 + 15x^2 + 12x - 6$
4	2	1	5	3	6	$2x^5 + x^4 - 14x^3 + 4x^2 + 10x - 2$
5	4	2	6	1	3	$4x^5 + 2x^4 - 28x^3 + 7x^2 + 18x - 2$
6	5	4	3	2	1	$-x$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_3(x) \to s_1(x)$
1	2	4	6	3	5	$x^5 - 2x^4 - 6x^3 - x^2 + 4x + 2$
2	4	5	3	1	6	$x^5 - 3x^4 - 3x^3 + 3x^2 - x$
3	1	2	5	6	4	$-x^5 + 3x^4 + 3x^3 - 3x^2 + 2x$
4	5	6	1	2	3	$-x^5 + 2x^4 + 7x^3 - 2x^2 - 7x + 1$
5	6	3	2	4	1	$x^5 - 4x^4 - x^3 + 9x^2 - 2x - 2$
6	3	1	4	5	2	$-x^5 + 4x^4 - 6x^2 + 4x$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_4(x) \to s_1(x)$
1	4	2	3	6	5	$-x^5 - 4x^4 + x^3 + 9x^2 + 2x - 2$
2	5	4	1	3	6	$x^5 + 4x^4 - 6x^2 - 4x$
3	2	1	6	5	4	$x^5 + 2x^4 - 7x^3 - 2x^2 + 7x + 1$
4	6	5	2	1	3	$x^5 + 3x^4 - 3x^3 - 3x^2 - 2x$
5	3	6	4	2	1	$-x^5 - 2x^4 + 6x^3 - x^2 - 4x + 2$
6	1	3	5	4	2	$-x^5 - 3x^4 + 3x^3 + 3x^2 + x$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_3(x) \to s_2(x)$
1	4	6	3	2	5	$x^5 - 4x^4 + 6x^2 - 4x$
2	1	4	5	3	6	$-x^5 + 4x^4 + x^3 - 9x^2 + 2x + 2$
3	2	1	6	5	4	$x^5 - 2x^4 - 7x^3 + 2x^2 + 7x - 1$
4	6	5	2	1	3	$x^5 - 3x^4 - 3x^3 + 3x^2 - 2x$
5	3	2	4	6	1	$-x^5 + 3x^4 + 3x^3 - 3x^2 + x$
6	5	3	1	4	2	$-x^5 + 2x^4 + 6x^3 + x^2 - 4x - 2$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_4(x) \to s_2(x)$
1	6	4	2	3	5	$x^5 + 3x^4 - 3x^3 - 3x^2 - x$
2	4	1	3	5	6	$x^5 + 2x^4 - 6x^3 + x^2 + 4x - 2$
3	1	2	5	6	4	$-x^5 - 3x^4 + 3x^3 + 3x^2 + 2x$
4	5	6	1	2	3	$-x^5 - 2x^4 + 7x^3 + 2x^2 - 7x - 1$
5	2	3	6	4	1	$-x^5 - 4x^4 + 6x^2 + 4x$
6	3	5	4	1	2	$x^5 + 4x^4 - x^3 - 9x^2 - 2x + 2$
b_1	b_2	b_3	b_4	b_5	b_6	Transforms from $s_4(x) \to s_3(x)$
1	3	2	5	4	6	$2x^5 + 6x^4 - 7x^3 - 8x^2 + 3x + 1$
2	6	3	1	5	4	$-x^4 - 2x^3 + 6x^2 - 2$
3	4	6	2	1	5	$2x^4 + 6x^3 - 7x^2 - 8x + 2$
4	1	5	6	3	2	$-x^3 - 3x^2 + 3x + 3$
5	2	1	4	6	3	$-2x^5 - 7x^4 + 4x^3 + 12x^2 + 3x - 2$
6	5	4	3	2	1	$-x$

the b_i given in these tables indicate the *order* of the roots needed to find the corresponding transformations. As before for Vandermonde's quintic, we assume the roots of the $s_i(x)$ to be ordered from the smallest to the largest. Thus, denoting by $s_1^{(i)}$, $i = 1, \ldots, 6$, the ordered roots of $s_1(x)$, the transformation shown in the first line of Table 2 is obtained when fixing $b_i = s_1^{(i)}$, and so on.

Similarly, as for the $s_i(x)$, polynomials characterized by totally real cyclic sextics and *second lowest* discriminant, namely 371, 293, are the following:

$$\begin{aligned}
t_1(x) &= x^6 - x^5 - 5x^4 + 4x^3 + 6x^2 - 3x - 1,\\
t_2(x) &= x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1,\\
t_3(x) &= x^6 - 3x^5 - 6x^4 + 4x^3 + 5x^2 - x - 1,\\
t_4(x) &= x^6 + 3x^5 - 6x^4 - 4x^3 + 5x^2 + x - 1,\\
t_5(x) &= x^6 - 2x^5 - 7x^4 + 6x^3 + 5x^2 - 5x + 1,\\
t_6(x) &= x^6 + 2x^5 - 7x^4 - 6x^3 + 5x^2 + 5x + 1,\\
t_7(x) &= x^6 - 5x^5 + 5x^4 + 6x^3 - 7x^2 - 2x + 1,\\
t_8(x) &= x^6 + 5x^5 + 5x^4 - 6x^3 - 7x^2 + 2x + 1.
\end{aligned}$$

As for the $s_i(s)$, note the conspicuous presence of two groups of four elements arising from the substitutions $x \to \pm x$ and $x \to \pm 1/x$ and suitable simplifications.

To each pair of polynomials $t_i(x)$ corresponds a set of six transformations analogous to the ones in Tables 2 and 3, resulting from similar root permutations. The total number of such transformations is 64, too many to be recorded here explicitly. However, the polynomials $t_i(x)$ allow them to be obtained easily if so desired, along with the proper root permutations leading to them.

3.4. *Equivalent totally real cyclic sextics with larger coefficients*

Table 4 reports isomorphisms having a direct bearing on the inner workings of the Hénon Hamiltonian repeller.[33] As seen in the previous section, sextics leading to *minimal discriminants* tend to have comparatively small coefficients. However, in real-life applications, the coefficients are not always so small, for instance the cluster $\mathbb{A}^{(5)}(x) = X(x)Y^2(x)Z^2(x)$ defining the orbital coordinates of six period-five trajectories

Table 4. Bridges among sextics arising in orbital clusters of the Hénon Hamiltonian repeller. Note the noninteger coefficients in the first column, not a common occurrence in the literature.

Direct	Inverse
$X \to Y$: $-\frac{1}{2}x^2 + 3$	$Y \to X$: $-x^4 + 2x^3 + 9x^2 - 13x - 16$
$X \to Z$: $-\frac{1}{4}x^4 + 3x^2 - x - 3$	$Z \to X$: $-x^4 - 2x^3 + 11x^2 + 11x - 30$
$Y \to Z$: $x^4 - 2x^3 - 10x^2 + 13x + 22$	$Z \to Y$: $-x^2 - x + 6 = -(x+3)(x-2)$
$U \to W$: $-\frac{1}{4}x^4 + 3x^2 - x - 3$	$W \to U$: $-4x^5 - 6x^4 + 63x^3 + 74x^2 - 234x - 219$
$-\frac{1}{2}x^2 + 3$	$2x^5 + 3x^4 - 32x^3 - 38x^2 + 121x + 116$

of the Hénon Hamiltonian repeller involves totally real sextics with larger coefficients:

$$X(x) = x^6 - 2x^5 - 14x^4 + 24x^3 + 32x^2 - 16x - 8, \quad \Delta_X = 2^{18} \cdot 31 \cdot 241 \cdot 389,$$
$$Y(x) = x^6 - 2x^5 - 16x^4 + 26x^3 + 81x^2 - 84x - 125, \quad \Delta_Y = 2^6 \cdot 31 \cdot 241 \cdot 389,$$
$$Z(x) = x^6 + 2x^5 - 16x^4 - 22x^3 + 85x^2 + 60x - 151, \quad \Delta_Z = 2^6 \cdot 31 \cdot 241 \cdot 389.$$

Similarly, a trio of period-five orbits is amalgamated into $\mathbb{B}^{(5)}(x) = U(x)W^2(x)$, defining the eighteen orbital coordinates as roots of a pair of totally real sextics,[33] with Galois group 6T7:

$$U(x) = x^6 - 22x^4 + 8x^3 + 124x^2 - 88x - 32, \quad \Delta_U = 2^{18} \cdot 3^4 \cdot 659^2,$$
$$W(x) = x^6 + 4x^5 - 12x^4 - 58x^3 + 12x^2 + 202x + 139, \quad \Delta_W = 2^6 \cdot 3^4 \cdot 659^2.$$

Table 4 shows that the orbits algebraically entangled together to form the above pair of orbital clusters have their coordinates related by *simple transformations* that, surprisingly, *allow back and forth passage among seemingly distinct orbits.* Unfortunately, the Galois group of $X(x)$, $Y(x)$, $Z(x)$, $U(x)$, and $W(x)$ is the symmetric group, meaning that these sextics cannot be solved in terms of a finite number of radical extractions and elementary arithmetic operations. But the transformations interconnecting these sextics show clearly that knowledge of just two sets of six roots is enough to interconnect in phase–space, all orbital points of the equations algebraically entangled in each cluster. The sextic trio of cluster $\mathbb{A}^{(5)}(x)$ is not isomorphic to the pair of sextics of cluster $\mathbb{B}^{(5)}(x)$. Table 4 contains transformations with non-integer coefficients, something that we have not been able to find in the literature.

3.5. *Equivalence among distinct families of isodiscriminant sextics*

As a quite remarkable final example, we consider a family of ten totally real sextics sharing the same discriminant, $810,448 = 2^2 \cdot 37^3$, but formed by two *nonisomorphic* families $f_i(x)$ and $g_i(x)$, defined in Eqs. (11)–(20). The four sextics $f_i(x)$ are isomorphic among themselves, as also are the six $g_i(x)$. However, none of the $f_i(x)$ is isomorphic to any of the $g_i(x)$ and vice-versa, despite the fact that they all share the same discriminant. In the notation of Butler and McKay[35] adopted by Maple, the Galois group of the $f_i(x)$ is 6T1, a cyclic semiabelian group, while the group of the $g_i(x)$ is 6T8, a solvable, semiabelian group.

$$f_1(x) = x^6 - 3x^5 - 2x^4 + 9x^3 - 5x + 1, \tag{11}$$
$$f_2(x) = x^6 + 3x^5 - 2x^4 - 9x^3 + 5x + 1, \tag{12}$$
$$f_3(x) = x^6 - 5x^5 + 9x^3 - 2x^2 - 3x + 1, \tag{13}$$
$$f_4(x) = x^6 + 5x^5 - 9x^3 - 2x^2 + 3x + 1, \tag{14}$$
$$g_1(x) = x^6 - 5x^5 + 8x^4 - 9x^3 + 8x^2 - 5x + 1, \tag{15}$$
$$g_2(x) = x^6 + 5x^5 + 8x^4 + 9x^3 + 8x^2 + 5x + 1, \tag{16}$$

Table 5. Some representative *bridges* that allow direct and inverse passage among polynomials of the families $f_i(x)$ and $g_i(x)$. There are no connections between the $f_i(x)$ and $g_i(x)$ and vice-versa, despite the fact that they all share the same discriminant $810,448 = 2^2 \cdot 37^3$.

Direct		Inverse	
$f_1 \to f_2$:	$-x$	$f_2 \to f_1$:	$-x$
	$x-1$		$x+1$
	$-x^5+2x^4+4x^3-6x^2-4x+2$		$-x^5-3x^4+2x^3+8x^2-x-2$
	$-x^5+3x^4+2x^3-8x^2-x+2$		$-x^5-2x^4+4x^3+6x^2-4x-2$
	$x^5-3x^4-2x^3+8x^2+x-3$		$x^5+2x^4-4x^3-6x^2+4x+3$
	$x^5-2x^4-4x^3+6x^2+4x-3$		$x^5+3x^4-2x^3-8x^2+x+3$
$g_1 \to g_2$:	$-x$	$g_2 \to g_1$:	$-x$
	$x^5-5x^4+8x^3-9x^2+8x-5$		$x^5+5x^4+8x^3+9x^2+8x+5$
$g_1 \to g_3$:	$-x^5+4x^4-4x^3+5x^2-3x+2$	$g_3 \to g_1$:	$-x^5+3x^4-6x^3+7x^2-2x$
	$x^5-4x^4+4x^3-5x^2+3x-1$		$x^5-2x^4+4x^3-3x^2-x+1$
$g_1 \to g_4$:	$-x^5+4x^4-4x^3+5x^2-3x+1$	$g_4 \to g_1$:	$-x^5-2x^4-4x^3-3x^2+x+1$
	$x^5-4x^4+4x^3-5x^2+3x-2$		$x^5+3x^4+6x^3+7x^2+2x$
$g_1 \to g_5$:	$-x$	$g_5 \to g_1$:	$-x$
	$x^5-5x^4+8x^3-9x^2+8x-4$		$x^5-2x^3+5x^2-x+2$
$g_1 \to g_6$:	$x+1$	$g_6 \to g_1$:	$x-1$
	$-x^5+5x^4-8x^3+9x^2-8x+4$		$-x^5+2x^3+5x^2+x+2$

$$g_3(x) = x^6 - 3x^5 + 6x^4 - 7x^3 + 2x^2 + x - 1, \tag{17}$$

$$g_4(x) = x^6 + 3x^5 + 6x^4 + 7x^3 + 2x^2 - x - 1, \tag{18}$$

$$g_5(x) = x^6 - x^5 - 2x^4 + 7x^3 - 6x^2 + 3x - 1, \tag{19}$$

$$g_6(x) = x^6 + x^5 - 2x^4 - 7x^3 - 6x^2 - 3x - 1. \tag{20}$$

Once again, note in Eqs. (11)–(20), two groups of four elements underlying the substitutions $x \to \pm x$ and $x \to \pm 1/x$, and the presence of two *outliers*, namely the reciprocal polynomials $g_1(x)$ and $g_2(x)$.

Proceeding as before, from the roots of Eqs. (11)–(20), one may easily obtain the large set of transformations allowing back and forth passage, local to the global, among both groups of sextics. There is a total of six transformations connecting each pair of $f_i(x)$ but just two transformations connecting pairs of $g_i(x)$. The complete set of transformations is omitted here, with just a few representative ones being given in Table 5.

4. Conclusions and Outlook

This paper has shown that Lagrange interpolation works as an efficient detector of equivalence and isomorphism among orbital equations of motion of algebraic dynamical systems governed by discrete-time mappings. This is a startling new application for a well-known interpolation technique of numerical analysis. Here, it is not used to *approximate* anything but, instead, as a means of obtaining *exact* *analytical expressions for isomorphisms*. We found polynomial interpolation to efficiently detect equivalences among equations of any signature, i.e. among polynomials having only real roots or not. The method is simple to implement and very fast. We anticipate polynomial interpolation to be a helpful tool to locate equivalences among the huge number[1] of orbital equations in systems of algebraic origin and polynomials in general. In particular, it should help to uncover equivalences among, e.g. the complicated *amalgamation* polynomial clusters arising in the Hamiltonian repeller limit of the Hénon map,[33] and among orbits of the Pincherle map, a paradigmatic map underlying the operating kernel of the so-called *chaotic computer*.[36–39]

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