Conjugacy classes and chiral doublets in the Hénon Hamiltonian repeller

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Abstract

Conjugacy classes hidden behind orbital symmetries under reflection are brought to light through the recognition of the fundamental role played by certain algebraic clusters underlying orbital equations of the area-preserving Hénon map, a discrete proxy of open Hamiltonian systems that exhibit chaotic scattering and transport. Specific number-fields ruling orbital coordinates are shown to slave large sets of orbits into being necessarily nonlinearly coupled. Particularly remarkable is the conjugacy class characterized by non-self-symmetric chiral doublets which completely dominates at higher periods.

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1. Introduction

A long-outstanding problem in classical mechanics and statistical physics, with far-reaching and important applications in several areas, is that connected with studying symmetries and asymmetries of periodic motions in phase-space. This problem was and still is one of the central problems and active fields of research in Celestial Mechanics as emphasized in very recent investigations of motions in planetary and satellite systems [1–4]. Apart from these technologically important applications, the study of symmetries and orbits in area- and orientation-preserving maps has also helped to clarify intricate dynamical processes like, for example, the emergence of infinitely many symmetric periodic orbits through saddle-node or equiperiod bifurcation [5] and the dynamics of open conservative systems, i.e., conservative systems having unbounded phase-space as, e.g., “leaking” Hamiltonian systems [6].

Here we study orbital symmetries and asymmetries computed analytically and systematically for the Hamiltonian (area-preserving) $b = -1$ limit of the Hénon map

$$x_{t+1} = a - x_t^2 + b x_{t-1}. \quad (1)$$

This map is a well-known discrete proxy of open Hamiltonian systems that exhibit chaotic scattering and transport [7]. It was found very early to reproduce experimental results involving ferromagnetic resonance in yttrium iron garnet [8].

Our key result is the discovery of a natural segregation of orbits into three conjugacy classes when one considers orbital symmetries with respect to a reflection in phase-space about the $y = x$ symmetry line, as defined in Table 4. As we argue below, under such reflection every periodic orbit (cycle) falls into one of three possible classes:

- **D**: diagonal cycles: self-conjugate periodic orbits with points on the diagonal, as illustrated in Fig. 1 below.
- **N**: non-diagonal cycles: self-conjugate periodic orbits without points on the diagonal, shown in Fig. 1 below.
- **C**: chiral doublets: pairs of non-self-conjugate cycles that map into each other, illustrated in Figs. 2–4 below.

This classification is independent of the control parameter $a$. Orbits are specially interesting for $a > a_h$, where $a_h \simeq 5.69931 \ldots$, since beyond this value there is a complete Smale horseshoe [9] and all orbits are real.

Formally, the conjugacy classes are defined by the three factors composing Eq. (5). The next section explains this equation and presents a methodology which allows factors to be computed systematically for arbitrary period.
2. The polynomials $P_k(x)$ and $S_k(\sigma)$

It is always possible to reduce the problem of finding the periodic orbits of dynamical systems of algebraic origin of any dimension to the much simpler problem of finding the roots of a fundamental pair of univariate polynomials, say,

$$P_k(x) \quad \text{and} \quad S_k(\sigma),$$

indexed by the orbital period $k$ and involving only the physical parameters of the model [10–12]. This pair of polynomials provides exact analytical expressions encoding simultaneously all $k$-periodic orbits as a function of the sum $\sigma$ of orbital points. We now summarize briefly the procedure employed to obtain the polynomials explicitly. The general procedure was introduced in Ref. [10] and applied in Refs. [11,12]. The generic origin of periodicity is discussed in Ref. [13].

First, from Eq. (1) note that a generic period-$k$ orbit is defined by a set $\{x_\ell\}$ of numbers containing $k$ coordinates $x_\ell$ of the orbit, which for convenience we label from $\ell = 1$ to $\ell = k$. These $k$ coordinates may be used to construct a polynomial of degree $k$ defining the orbit

$$P_k(x) = \prod_{\ell=1}^{k} (x - x_\ell).$$

Now, recall a basic text-book result about the relation between roots and coefficients of polynomials: in any polynomial of degree $k$, the negative of the coefficient of the term of degree $k - 1$ is the elementary symmetric function representing the sum of all roots, in our case, the sum of all orbital coordinates $x_\ell$. We call this sum $\sigma$ and, by eliminations with the help of Eq. (1), express all other coefficients of $x$ in $P_k(x)$ as functions of $\sigma$. After all eliminations $P_k(x)$ is a polynomial involving three quantities only: $x$, $\sigma$ and the control parameter $a$. For explicit examples, see Eqs. (6), (15) and (16).

The process of replacing all coefficients of $P_k(x)$ by functions of $\sigma$ produces an important additional polynomial, a constraint equation for $\sigma$, denoted by

$$S_k(\sigma) = 0.$$

This polynomial involves only $\sigma$ and the parameters of the model, $a$ in the present case. For explicit examples, see Table 1 and, more generically, Eq. (5). The roots $\sigma_\ell$ of Eq. (4) are the numerical values of $\sigma$ needed in the corresponding polynomial $P_k(x)$ to fix each individual orbit.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$S_k(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$D_1(\sigma) = \sigma^2 + 2\sigma - a$</td>
</tr>
<tr>
<td>2</td>
<td>$N_2(\sigma) = \sigma - 2$</td>
</tr>
<tr>
<td>3</td>
<td>$D_3(\sigma) = \sigma^2 - 2\sigma + 2 - a$</td>
</tr>
<tr>
<td>4</td>
<td>$D_4(\sigma) = \sigma$, $N_4(\sigma) = \sigma^2 - 4\sigma$</td>
</tr>
<tr>
<td>5</td>
<td>$D_5(\sigma) = \sigma^6 - 2\sigma^5 - (11a + 12)a^4 + 12(2 + 3a)\sigma^3$ + $(20a + 36 + 19a^2)\sigma^2 - 2(a + 6)(17a + 2)\sigma - 9a^3$ + $88a^2 + 56 + 28a$</td>
</tr>
</tbody>
</table>

We remark that our choice of expressing all coefficients of $P_k(x)$ as functions of the sum $\sigma$ of orbital points is motivated by the interpretation of $\sigma$ as a sort of mean-field signature of each individual orbit. As pointed out in Ref. [10], any coefficient of $P_k(x)$ could be used to represent all remaining coefficients equally well, since all coefficients are interconnected by the well-known elementary symmetric functions considered first by Girard and by Newton [14], constructed with the roots of $P_k(x)$. Two particularly fruitful parameterizations are $\sigma$, the sum of the orbital points, and $\pi$, their product. Here we focus only on $\sigma$.

In the next section we show that $S_k(\sigma)$ is formed by certain characteristic factors and discuss their distinct nature. As indicated in Table 1, up to period $k = 5$ the polynomial $S_k(\sigma)$ contains only two factors of type, namely $D_k(\sigma)$ and $N_k(\sigma)$. The meaning of this notation is also explained in the next section.

3. The factors $C_k$, $D_k$, $N_k$ and the classes

The purpose of this section is to show that the polynomials $S_k(\sigma)$ of Eq. (4) emerge automatically decomposed into three factors, namely,

$$S_k(\sigma) = C_k(\sigma) D_k(\sigma) N_k(\sigma),$$

and that these factors are the mathematical objects characterizing the clusters responsible for the three algebraic conjugacy classes. The fact that the roots $\sigma_\ell$ originating from each cluster are number-theoretically conjugate quantities justifies the name conjugacy classes. From an algebraic point of view, this conjugacy is the key theoretical novelty. It implies strong orbital correlations and interdependencies.

To fix ideas, consider the problem of determining explicitly the exact analytical expressions for all possible orbits of period $k = 4$ of the Hénon map, Eq. (1). The solution of this problem is trivial because Ref. [10] contains the solution for arbitrary $a$ and $b$. Substituting $b = -1$ in the general expressions one obtains the polynomial $P_4(x) = P_4(x, \sigma)$ which, through the parameter $\sigma$, encodes simultaneously all possible period-4 orbits:

$$P_4(x) = x^4 - \sigma x^3 + (\sigma^2 + 2\sigma - 4a)x^2/2$$
$$- (\sigma^2 + 6\sigma - 10a)\sigma x/6 + \sigma^4/24$$
$$+ \sigma^3/2 - 2a\sigma^2/3 + a^2 - a\sigma.$$

In the present case [10], $\sigma$ is a root of the cubic

$$S_4(\sigma) = \sigma (\sigma^2 - 4a).$$

The degree of $S_4(\sigma)$ tells that Eq. (1) contains altogether three period-4 orbits, one for each root of $S_4(\sigma)$, namely $-2r$, $0$, $2r$, where $r = \sqrt{a}$. When substituted into $P_4(x)$, these roots generate three orbits, respectively,

$$P_{4,1}(x) = (x^2 - 2a)(x + r)^2,$$
$$P_{4,3}(x) = x^4 - 2ax^2 + a^2 = (x^2 - a)^2,$$
$$P_{4,7}(x) = (x^2 - 2a)(x - r)^2.$$
These orbits are shown in Fig. 1 for \( a = 6 \), the smallest integer where a complete Smale horseshoe exists [12]. All three orbits in Fig. 1 are clearly self-symmetric under reflection about the \( y = x \) symmetry diagonal and come in two flavors: with or without points on the axis. Obviously, self-symmetric orbits with even periods \( k \) cannot have an odd number of points on the diagonal. Self-symmetric orbits may have points on the symmetry diagonal or on the time-reversal symmetry parabolas shown in the figure.

Self-symmetric orbits are the only ones observed up to period 5 and Table 1 summarizes the specific expressions \( S_k(\sigma) \) for \( k \leq 5 \). The simple two-factor structure of \( S_k(\sigma) \) found for periods \( k \leq 5 \) changes when \( k \geq 6 \) by the appearance of an additional factor, \( C_k(\sigma) \), containing all chiral doublets, as we now show.

For arbitrary values of \( a \) explicit calculations show that there are nine period-6 orbits and that

\[
S_6(\sigma) = C_6^2(\sigma)D_6(\sigma)N_6(\sigma),
\]

where

\[
C_6(\sigma) = \sigma - 2,
\]

\[
D_6(\sigma) = \sigma^2 + 4\sigma - 4a,
\]

\[
N_6(\sigma) = \sigma^2 + 2\sigma^4 - 4(5a + 4)\sigma^3 + 8a^2\sigma
\]

\[
+ 4(16a^2 + 12a + 9)\sigma + 128a^2 - 96a + 72.
\]

As before for \( P_4(x) \), in the Hamiltonian case \( b = -1 \) and for \( \sigma = 2 \) a general expression available in the literature [11] simplifies to

\[
P_6(x) = \left[ x^3 - (1 + \tau)x^2 - ax + a(1 + \tau) - 1 \right]
\]

\[
\times \left[ x^3 - (1 - \tau)x^2 - ax + a(1 - \tau) - 1 \right],
\]

where \( \tau = \sqrt{a - 3} \) is the relative-quadratic irrationality breaking the sextic generically. A pitchfork bifurcation occurs for \( a = 3 \), when a stable class \( D \) orbit looses stability in favor of a chiral doublet, stable for \( 3 < a < 3.00795 \). ... The roots of Eq. (15) define the coordinates of the chiral doublet shown in Fig. 2. Individual orbits are obtained by selecting the proper in-phase combination of roots needed to start iterating Eq. (1). The stability of the doublet is ruled by degenerate \( \sigma \) values, identical for both orbits. Thus, chiral doublets are always both stable or both unstable, depending on \( a \). The factors \( C_k(\sigma) \) of \( S_k(\sigma) \) define the specific values of \( \sigma \) leading to chiral doublets.

Analogously, for period 7 we compute all 18 orbits and a decomposition \( S_7(\sigma) = C_7^2(\sigma)D_7(\sigma) \) where \( D_7(\sigma) \) is a polynomial of degree 14 and \( C_7(\sigma) = \sigma^2 - 2\sigma - a \). Thus, there are...
two chiral doublets and 14 self-symmetric orbits with points on the diagonal. The roots $\sigma_i$ of $C_7(\sigma)$ generate a pair of chiral doublets when substituted in

$$\tilde{P}_7(x) = x^7 - 3x^6 - (3a - 2\sigma)x^5 - (2a - (3a - 4)\sigma - 4)x^4$$
$$+ (3a^2 - 2(2a - 1)\sigma + 1)x^3$$
$$+ (4a^2 - 10a - (3a^2 - 8\sigma + 1)\sigma - 2)x^2$$
$$- (a - 1)(a^2 - 2a\sigma + a + 2)x$$
$$- 2a^3 + 6a^2 + 2a + 3(a^3 - 4a^2 + a - 2)\sigma.$$

One doublet is obtained for

$$\sigma_{11} = \sigma_{13} = 1 + \sqrt{a + 1},$$

while the conjugate doublet follows from the conjugate $\sigma$, namely,

$$\sigma_{23} = \sigma_{29} = 1 - \sqrt{a + 1}. \quad (18)$$

Fig. 3 shows the orbits composing the period-7 doublets. Note that all four orbits are compactly encoded by $\sigma$ in a single polynomial.

For period 8 we obtain the decomposition $S_8(\sigma) = C_2(\sigma) \times D_8(\sigma) \times N_8(\sigma)$, defining 6 orbits of class $D$, 12 of class $N$, and 6 chiral doublets defined by

$$C_8(\sigma) = \sigma^6 - 4\sigma^5 - 4(2a + 1)\sigma^4 + 16(2a + 1)\sigma^3$$
$$+ 16a(a + 1)\sigma^2 - 64a(a + 1)\sigma - 16. \quad (19)$$

The polynomial $W_8(x)$ encoding simultaneously all period-8 doublets is given in Appendix A.

Fig. 4 shows the six period-8 doublets, two of them similar to those in Fig. 3 but with an extra orbital point. Of course, there are also period-8 orbits of class $D$ and $N$, but the polynomials defining them are too big to be listed here.

The systematic seems now well-established and was corroborated by computing with ad hoc computer algebra routines all pairs $P_k(x)$ and $S_k(\sigma)$ up to period $k = 20$. Again, and for arbitrary values of the control parameter $a$, $S_k(\sigma)$ always decomposes systematically as represented in Eq. (5). The specific number of orbits found in each class is summarized for $k \leq 20$ in Table 2.

From the results above one sees that the roots of the factors $C_k$, $D_k$, $N_k$ of $S_k(\sigma)$ define the orbits in each class, $C$ standing for chiral, $D$ for diagonal and $N$ for non-diagonal orbits. Symmetry under reflection about the $y = x$ diagonal forbids the existence of class $N$ for odd $k$, i.e., $N_k(\sigma) \equiv 1$ for odd $k$. Symmetry also imposes that for even (odd) periods class $D$ orbits will have an even (odd) number of points on the diagonal. We can find no reason for the classes not to remain well-defined as the period further grows. The specific number-fields (base fields) defined by the $\sigma$-factors in Eq. (5)
are the strong links slaving orbits into clusters and sums of orbital points into classes.

It is known that in the limit in which all possible horseshoe orbits exist, binary Smale horseshoe labels are useful because they assign a unique name to each periodic point and facilitate counting cycles. For instance, the number $M_k$ of cycles of period $k$ is related to the $2^k$ cycle points by the Möbius inversion formula

$$M_k = \frac{1}{k} \sum_{d|k} \mu(k/d)2^d,$$

(20)

where $\mu(n)$ is the Möbius function [15], and the sum runs over all divisors $d$ of $k$. Table 2 summarizes the partition of the $M_k$ orbits as obtained by computing all of them explicitly. For prime $k$ there are $2^{(k+1)/2} - 2$ orbits of class $D$. The partition of $M_k$ into the number of orbits in each class is a nice challenge in combinatorial analysis.

The decomposition summarized by Eq. (5) and the overwhelming abundance of “handedness” manifest in Table 2 are our main theoretical results. They are both independent of the numerical value of the control parameter $a$. The remarkably strong dominance of chiral (asymmetric) orbits seen in Table 2 as the period grows contrasts sharply with recent remarkable findings concerning the signature of time-reversal symmetry of polynomial automorphisms over finite fields, an exceptional context where symmetric orbits dominate statistics over asymmetric ones [16].

4. Final comments

The investigation of conservative dynamical systems with two degrees of freedom has a long history [17–24]. Birkhoff [18,19] observed that conservative dynamical systems with two degrees of freedom may be decomposed generically into a pair of involutions. These involutions were first exploited by de Vogelaere [20]. For instance, the reversing involution, or reversor, $R(x, y) = (y, x)$, may be conveniently used to segregate orbits into two broad classes: symmetric and non-symmetric orbits. Until now the emphasis has been by far in the study of symmetric orbits, the easiest ones to find numerically. But non-symmetric orbits have been also occasionally mentioned in the literature, mostly rather briefly. For instance, after summarizing de Vogelaere’s classification [20] of symmetric periodic orbits based on the involution $R$, Devaney [21] remarks on p. 96 that “The remaining [orbits] are called non-symmetric. They always occur in pairs $\gamma$ and $R(\gamma)$. A whole section dedicated to asymmetric orbits appears in the very nice paper by Jiménez-Lara and Piña [23] dealing with Störmer’s problem, namely the problem of finding the orbits for an electric charge in a magnetic dipole field [25].

The general non-self-symmetric involutions and chirality (mirror symmetry of non-self-symmetric orbital pairs) discussed here do not seem to have been observed before. These involutions are different from the popular involutions normally connected with time-reversal symmetries and should not be confused with them. The connection between involutions and factorizations is the subject of a separate work [29]. Generically, factorization is a very rare event in algebraic geometry and in this sense the orbital segregation summarized by Eq. (5) is rather surprising.

Concerning generality, we recall a result of Friedland and Milnor stating that every polynomial automorphism that is not dynamically trivial is conjugate to a composition of generalized Hénon transformations [30]. Thus, we believe that the results reported here could be also valid for physical systems ruled by rational equations of motion, polynomials in particular. This, however, remains to be investigated more closely. The study of planar polynomial automorphisms whose inverse is also a polynomial map, the situation discussed here, is of great interest for applications in physics since reversibility is frequently associated with Hamiltonian systems and, more generally, to conservative dynamics. Another interesting open question is the possible application of the algebraic conjugacy of orbits to the effective computation of the topological entropy, to assess the complexity of Hénon and Hénon-like systems along the lines of recent works [31–33].

From a physical point of view, Eq. (5) may be regarded as implying two different levels of orbital organization. First, there is a sort of coarse-grained macroscopic upper level of orbital organization manifest by the decomposition of $S_k(\sigma)$ into the classes $C$, $D$, and $N$. Then, underlying each individual class there is a microscopic level responsible for the selection of highly symmetric number-field conjugacies and very delicate balance between them, manifest in the coordinates of orbital points of individual orbit. This latter level is the lowest level possible from a mathematical point of view: it is the “atomic” number-theoretic level responsible for exquisite symmetries and conjugacies between the individual numbers defining orbital coordinates.

So far, symmetric orbits have overwhelmingly dominated attention in the literature. From Table 2 it is possible to get the impression that the rarity of asymmetric orbits at low periods could explain their relative oblivion. This work contributes to reestablish the balance by shedding light on the crowded sets of asymmetric orbits, by far the dominating class of orbits at higher periods. At any rate, symmetric orbits are extremely useful but quite exceptional.

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1. We are indebted to C. Mira for kindly pointing out that our involutions may be described in a generic symbolic plane with the help of the one-dimensional quadratic map, as done in Section 6.6.2 of his book [26]. See also work by Kawakami [27,28].
Appendix A

This appendix records the polynomial cluster defining the exact orbital coordinates simultaneously for all six chiral doublets in Fig. 4. This polynomial results from straightforward but very long computations and is given in Table 3. It contains many number-theoretical features, in particular it is composed by relatively big primes.

For arbitrary values of \(a\), the 12 period-8 chiral orbits are entangled among the roots of the following polynomial cluster

\[
W_8(x) = x^{48} - 4x^{47} - 12(2a - 1)x^{46} + 4(23a - 2)x^{45}
+ 6(46a^2 - 48a - 3)x^{44} - 4(253a^2 - 60a - 27)x^{43}
- 4(506a^3 - 825a^2 - 43a + 34)x^{42}
+ 4(1771a^3 - 798a^2 - 519a + 21)x^{41} + \ldots
+ a^{20} - 18a^{19} + 117a^{18} - 267a^{17} - 394a^{16}
+ 3017a^{15} - 4328a^{14} - 1634a^{13} + 8228a^{12}
- 3736a^{11} - 3116a^{10} + 620a^9 + 2112a^8 + 2104a^7
- 3924a^6 + 380a^5 + 328a^4 + 540a^3 - 16a^2
- 128a + 16)x^3
+ a^{24} - 24a^{23} + 238a^{22} - 1236a^{21} + 3404a^{20}
- 3672a^{19} - 408a^{18} + 13880a^{17} - 5156a^{16}
- 14376a^{15} + 10740a^{14} + 1944a^{13} + 6112a^{12}
- 8144a^{11} - 4776a^{10} + 5168a^9 + 16a^8 - 112a^7
+ 464a^6 - 512a^5 + 480a^4 - 352a^3 + 48a^2 - 32a + 16.
\]

(A.1)

These coefficients are the key to the divisibility of individual orbits. The complete polynomial requires about two extra pages to be recorded and is therefore omitted. It is however available from the authors on request. We just mention that this and similar polynomials for higher periods represent parameterized functions having cyclic Galois group and an exceptionally high degree of symmetries and invariances.

Table 4 indicates for \(a = 6\) the proper in-phase sequence of roots needed to iterate Eq. (1) and obtain the orbits in Fig. 4. The 12 period-8 orbits are defined by the roots of

\[
W_8(x) = \sum_{i=0}^{48} h_i x^i,
\]

(A.2)

with \(h_i\) being listed in Table 3 and pairs of roots in Table 4.

References