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ScienceDirect
Physics Letters A 360 (2007) 512–514
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Counting orbits in conjugacy classes of the Hénon Hamiltonian repeller

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Received 8 May 2006; accepted 28 August 2006

Available online 5 September 2006

Communicated by A.R. Bishop

Dedicated to Prof. Carles Simó, Barcelona, on the occasion of his 60th birthday

Abstract

We present exact formulas solving the problem of partitioning the total number M_k of period-k orbits of the area-preserving Hénon map into the number of orbits building its three possible conjugacy classes. The formulas are valid for any arbitrary period k. They are derived with combinatorial methods and constitute a nice application of the number-theoretic Möbius inversion formula to a key problem in physics and dynamical systems. A handy MAPLE implementation of the formulas is also provided.

A number of orbital symmetries and asymmetries computed analytically and systematically for the Hamiltonian (area-preserving) $b = -1$ limit of the Hénon map

$$(x, y) \mapsto (a - x^2 + by, x) \tag{1}$$

have been studied recently in this Journal [1,2]. The key result reported was the existence of a natural segregation of all orbits into three *algebraic conjugacy classes* with respect to a spatial reflection $\mathbf{R}(x, y) = (y, x)$ about the $y = x$ symmetry diagonal in phase-space. Under reflection, every periodic orbit (cycle) was found to fall into one of three classes: (i) diagonal class $\mathcal{D}$, formed by self-symmetric orbits *with* points on the $y = x$ symmetry diagonal, (ii) non-diagonal class $\mathcal{N}$, formed by self-symmetric orbits *without* points on symmetry diagonal, and the (iii) chiral class $\mathcal{C}$, formed by *pairs* of non-self-symmetric cycles that map into each other.

Each class contains a characteristic algebraic signature embodied by a specific orbital decompositions (factorizations) [1]. To find a bridge connecting symmetries under reflection with algebraic factorization (segregation) of the equations of motion is rather surprising. The orbital segregation is independent of the control parameter a and is specially interesting for $a > a_h \simeq 5.69931\ldots$, the value beyond which there is a complete Smale horseshoe and all orbits are real [1,2]. In this context, a central problem is to count the number C_k, D_k, N_k of orbits in each class. In Ref. [1] these numbers were determined empirically by manually counting a total of 111,013 orbital points explicitly, for all periods up to $k = 20$. The aim of this Letter is to report exact analytical expressions, Eqs. (3)–(7), that solve the problem for good, for any arbitrary period k.

The problem of counting periodic orbits and its partitions is among the first problems that one needs to address [3–9]. For the paradigmatic quadratic map it was addressed very early by Myrberg, in what appears to be one of the first applications of computers to dynamics [10,11]. Apart from counting orbits, he knew well how to exploit symbolic dynamics and what was later named “itineraries” and “kneading sequences” [12] to efficiently tabulate parameters with no less than 11 digits of accuracy [10]. The problem of counting orbits for the Hénon map was also addressed very early, in a pioneering work by Simó [13] using an approach centered in the strange attractor creation/destruction. While all orbits exist for all parameters values, many orbits live in the complex sector of the phase space, as mentioned. The generic problem of orbital stability for two-dimensional mappings was considered by Bountis and Helleman [14]. For an interesting account of the history of dynamical systems and chaos and its cultures see the paper by Aubin and Dalmedico [15].

The direct combinatorial problem of determining the partitions C_k, D_k, N_k individually seems to be very hard. However

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.08.065

there is an efficient way of getting indirectly to them by counting the orbital points lying on symmetry axis of the problem. This is what we do. The approach is a nice application of enumerative combinatorics and the number-theoretic Möbius inversion formula to a key problem in physics and dynamical systems. Several complementary aspects of combinatorial dynamics are discussed in Ref. [16].

In the limit when all possible horseshoe orbits exist, binary Smale horseshoe labels assign a unique tag to each periodic point and facilitate counting cycles. Thus, the number M_k of cycles of period k is related to the 2^k cycle points by the Möbius inversion formula

$$M_k = \frac{1}{k}\sum_{d|k}\mu(k/d)2^d, \tag{2}$$

where $\mu(n)$ is the Möbius function [17], and the sum runs over all divisors d of the period k.

Alternatively, by considering how the orbital points organize themselves in the "2D orderings" introduced in Ref. [2], it is not difficult to find that the number A_k of *axial* points (i.e., of points lying on the $y = x$ axis) and the number Q_k of points lying on the *quadratic* symmetry parabolas are, respectively,

$$A_k = \sum_{d|k}\mu(k/d)2^{\lfloor (d+1)/2\rfloor}, \tag{3}$$

$$Q_k = \sum_{d|k}\mu(k/d)2^{\lfloor (d+2)/2\rfloor}, \tag{4}$$

where the symbol $\lfloor\alpha\rfloor$ denotes the integer part of α. The notation above was chosen so as not to conflict but to complement that in Refs. [1,2]. The approach used to get A_k and Q_k is similar to one used to count terms in the densities and fluxes of the Korteweg–de Vries and modified Korteweg–de Vries equation using Ferrers diagrams [18].

From A_k and Q_k it is easy to get the exact partitions of M_k into the number C_k, D_k, N_k of orbits in each class. All that needs to be done is to take into account the appropriate symmetry of the "boundary conditions" imposed by the alternating odd/even nature of the period k as it grows. Thus,

$$D_k = \begin{cases} A_k & \text{if } k \bmod 2 = 1, \\ A_k/2 & \text{otherwise,} \end{cases} \tag{5}$$

and

$$N_k = \begin{cases} 0 & \text{if } k \bmod 2 = 1, \\ Q_k/2 & \text{otherwise.} \end{cases} \tag{6}$$

The number C_k of chiral orbits is then

$$C_k = M_k - D_k - N_k. \tag{7}$$

Note that the possibility of expressing the orbital growth analytically in simple closed form for physically meaningful subsets of the total set of possible orbits is a rare event.

Table 1 illustrates the partition of M_k as computed with the above formulas using the program listed in Appendix A.[1] For a period as low as $k = 30$ there are about 36×10^6 orbits, virtually all of them being asymmetric (chiral doublets). These numbers reproduce those in Table 2 of Ref. [1], which were obtained numerically by computing orbits explicitly. For prime k there are $D_k = 2^{(k+1)/2} - 2$ orbits in class $\mathcal{D}$. Furthermore, for even periods k (i.e. whenever $N_k \neq 0$) the ratio N_k/D_k displays two intercalated behaviors as the period grows: (i) a period-4 alternation: $N_{4i}/D_{4i} \equiv 2$ for $i = 1, 2, \ldots$, and (ii) a steady convergence to 2:

$$\lim_{i\to\infty}\frac{N_{4i+2}}{D_{4i+2}} \to 2. \tag{8}$$

The *direct* partition of M_k into the number of orbits in each class seems to remain a nice challenge in combinatorial analysis.

A difficult but very important open problem for applications is to obtain general formulas disclosing the usually exceptional regular organization of the orbits in parameter space. For instance, to discover why stability islands nucleate so regularly in parameter space of, say, popular CO_2 laser models [20], in paradigmatic models involving non-Cauchy–Riemann (non-conformal) maps without the familiar critical points [21] or in much more general frameworks [22].

Table 1
Partitions of the total number M_k of period-k orbits, defined in Eq. (2), into the number of orbits composing each conjugacy class. The rightmost column shows the percentage of chiral orbits. Note that we list $C_k/2$, not C_k

k	M_k	$C_k/2$	D_k	N_k	%
1	2	0	2	0	0
2	1	0	0	1	0
3	2	0	2	0	0
4	3	0	1	2	0
5	6	0	6	0	0
6	9	1	2	5	22.2
7	18	2	14	0	22.2
8	30	6	6	12	40.0
9	56	14	28	0	50.0
10	99	30	12	27	60.6
11	186	62	62	0	66.7
12	335	127	27	54	75.8
13	630	252	126	0	80.0
14	1161	500	56	119	84.9
15	2182	968	246	0	88.7
20	52377	25446	495	990	97.16
25	1342176	666996	8184	0	99.39
30	35790267	17870709	16242	32607	99.86
35	981706806	490722342	262122	0	99.97

Acknowledgements

The author is grateful to Prof. C. Simó for the motivation to obtain the exact expressions reported here. This work is supported by the Air Force Office of Scientific Research and by the Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil.

[1] The author is indebted to R.S. MacKay for kindly pointing out that the numbers in Table 1 up to $k = 12$ are among those computed by hand for a different normal form and listed in Table 1.2.3.5.1 of his PhD thesis, Princeton, 1982, reprinted in Ref. [19], both unavailable to us.

Appendix A

Here is a simple MAPLE program for the automatic partition of M_k into the number of orbits C_k, D_k, N_k contained in each class.

```
with (numtheory);
for k  from 1 to 35 do
zk[k]:= 0:   ak[k]:= 0:   qk[k]:= 0:
for d in divisors(k) do
zk[k]:= zk[k]+ mobius(k/d)*2^d:
ak[k]:= ak[k]
        + mobius(k/d)*2^trunc((d+1)/2):
qk[k]:= qk[k]
        + mobius(k/d)*2^trunc((d+2)/2):
  od:
mk[k]:= zk[k]/k:
dk[k]:= `if` (k mod 2 = 1,ak[k],ak[k]/2):
nk[k]:= `if` (k mod 2 = 1, 0,qk[k]/2):
ck[k]:= mk[k] - dk[k] - nk[k]:
ratio:= 100.0*ck[k]/mk[k]:
if (k mod 1 = 0) then
print(k,mk[k],ck[k],ck[k]/2,
              dk[k],nk[k],ratio):
end if: od:
```

References

[1] A. Endler, J.A.C. Gallas, Phys. Lett. A 356 (2006) 1.
[2] A. Endler, J.A.C. Gallas, Phys. Lett. A 352 (2006) 124.
[3] E. Piña, Phys. Rev. A 34 (1986) 574;
O. Chavoya-Aceves, F. Angulo-Brown, E. Piña, Physica D 14 (1985) 374.
[4] K.M. Brucks, Adv. Appl. Math. 8 (1987) 434.
[5] M. Lutzky, Phys. Lett. A 131 (1988) 248;
M. Lutzky, Phys. Lett. A 177 (1993) 338.
[6] R.L. Bivins, J.D. Louck, N. Metropolis, M.L. Stein, Physica D 51 (1991) 3.
[7] F. Xie, B. Hao, Physica A 202 (1994) 237.
[8] W.Y.C. Chen, J.D. Louck, Adv. Appl. Math. 18 (1997) 18.
[9] A. Bridy, R.A. Perez, Nonlinearity 18 (2005) 1945.
[10] P.J. Myrberg, Ann. Acad. Sci. Fenn., Ser. A 256 (1958) 1;
P.J. Myrberg, Ann. Acad. Sci. Fenn., Ser. A 268 (1959) 1;
P.J. Myrberg, Ann. Acad. Sci. Fenn., Ser. A 336 (1963) 1.
[11] P.J. Myrberg, J. Math. Pures Appl. 41 (1962) 339.
[12] J. Milnor, W. Thurston, in: J.C. Alexander (Ed.), Dynamical Systems, in: Lecture Notes in Mathematics, vol. 1342, Springer-Verlag, New York, 1987.
[13] C. Simó, J. Stat. Phys. 21 (1979) 465.
[14] T. Bountis, R. Helleman, J. Math. Phys. 22 (1981) 1867;
T. Bountis, Physica D 3 (1981) 577.
[15] D. Aubin, A.D. Dalmedico, Hist. Math. 29 (2002) 273.
[16] L. Alsedà, J. Llibre, M. Misiurewicz, Combinatorial Dynamics and Entropy in Dimension One, World Scientific, Singapore, 1993.
[17] G.E. Andrews, Number Theory, Dover, New York, 1994, Section 6-4, reprint of the 1971 original.
[18] M.T. Fontenelle, J.A.C. Gallas, Comput. Phys. Commun. 60 (1990) 225.
[19] R.S. MacKay, Renormalisation in Area-Preserving Maps, PhD thesis, Princeton, 1982, reprinted in Advanced Series in Nonlinear Dynamics, vol. 6, World Scientific, Singapore, 1993.
[20] C. Bonatto, J.C. Garreau, J.A.C. Gallas, Phys. Rev. Lett. 95 (2005) 143905;
J.A.C. Gallas, Physica A 283 (2001) 17;
J.A.C. Gallas, Appl. Phys. B 60 (1995) S-203;
J.A.C. Gallas, Physica A 202 (1994) 196;
J.A.C. Gallas, Phys. Rev. Lett. 70 (1993) 2714.
[21] A. Endler, J.A.C. Gallas, C. R. Acad. Sci. (Paris) Ser. I 342 (2006) 681.
[22] J.A.C. Gallas, physics/0604097.