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Reductions and simplifications of orbital sums in a Hamiltonian repeller

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Abstract

We solve the "center of mass puzzle", namely the empirical observation that in the presence of a complete Smale horseshoe, the sum of the coordinates of orbital points for low periodic orbits of the Hamiltonian Hénon map reduces to simple rational numbers every so often. We show that such reductions occur systematically in phase-space and arise from specific factorizations of the equations defining orbital coordinates and their sums. The factorizations manifest sets of correlated orbital clusterings and imply an algebraic way of ordering orbits.

1. Introduction

During the numerical computation of sets of periodic orbits typically necessary for the evaluation of spectra of evolution operators of the Hamiltonian (area preserving) Hénon map,

$$y_{t+1} = 1 - a y_t^2 - y_{t-1}, \tag{1}$$

Vattay [1] noticed that for $a = 6$ the sum σ of periodic points of some cycles unexpectedly reduced to very simple numbers, the *integers* which appear in the column "σ" in Table 1 below. Such sudden reductions of σ were referred to as the "center of mass puzzle" [2] and a call for an explanation of this phenomenon was issued by Cvitanović et al. [3]. An explanation of such reductions is important because analogous reductions should show up in phase spaces of more general physical systems modeled by polynomial mappings.

The purpose of this Letter is to present an arithmetical explanation of the center of mass puzzle. We show that simple sums emerge whenever certain factorizations are possible. These factorizations arise from simplifications among the specific number fields [4,5] underlying the dynamics. To see how such factorizations arise one first needs to obtain *exact* analytical expressions for the sums of coordinates of orbital points.

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2. Parameterized orbital equations

The standard procedure to derive exact analytical expressions for the orbital sums σ involves computing resultants. The trouble is that automatic computations based on Grobner basis and commercial softwares do not work yet beyond period 4 and, therefore, cannot be used to address the smallest non-trivial periods 5 and 6 considered numerically by Vattay et al. [1]. To bypass this difficulty we use an efficient shortcut for the computation of resultants [6–8] which effectively reduces multidimensional systems to much simpler *one-dimensional equivalent systems*[1] which can be analyzed for much higher periods. For instance, we derived analytical results up to period 11 although here we focus only in the periods considered in Refs. [1,2], namely periods not higher than 6.

As discussed recently [6,7], for a given period k, algebraic elimination yields a reduced polynomial $P_k(x) \equiv P_k(x;\sigma)$ of degree k parameterized by σ, the sum of the k periodic points x_i, $i = 1, \ldots, k$ in a given periodic orbit, where x denotes the variable of the one-dimensional equivalent system. The benefits of $P_k(x)$ are (i) its k roots x_i define the k periodic points,

[1] The reduction mentioned here corresponds to the elimination of all but one variable from the multidimensional equations of motion. This may always be achieved computing resultants [9]. This reduction should not be confused with standard expressions like Eq. (2), which remain multidimensional despite the use of a single variable.

Table 1
Sums σ for $a = 6$ and periods $k \leqslant 6$. The factorization classes manifest remarkably correlated orbital clusterings and imply a new ordering. The constants α, β, etc., are defined later in the Letter

k	Coding		σ	Degree of σ	Class
	Binary	Decimal			
1	0	1,0	**3.6457**	2	$D(\sqrt{7})$
	1	1,1	**1.6457**	2	$D(\sqrt{7})$
2	01	2,1	2	1	$N(\sqrt{3})$
3	001	3,1	**−1.2360**	2	$D(\sqrt{5})$
	011	3,3	**3.2360**	2	$D(\sqrt{5})$
4	0011	4,3	0	1	$D(\sqrt{6})$
	0001	4,1	**−4.8989**	2	$N(\alpha)$
	0111	4,7	**4.8989**	2	$N(\beta)$
5	00001	5,1	−8.5561	6	$D(\chi)$
	00011	5,3	−3.6399	6	$D(\chi)$
	00101	5,5	0.9082	6	$D(\chi)$
	00111	5,7	1.4907	6	$D(\chi)$
	01011	5,11	5.2241	6	$D(\chi)$
	01111	5,15	6.5729	6	$D(\chi)$
6	001011	6,11	2	1	$C(\varphi)$
	001101	6,13	2	1	$C(\varphi)$
	000011	6,3	**−7.2915**	2	$D(\sqrt{7})$
	001111	6,15	**3.2915**	2	$D(\sqrt{7})$
	000111	6,7	**−2.1961**	2	$N(\sqrt{3})$
	011111	6,31	**8.1961**	2	$N(\sqrt{3})$
	000001	6,1	−12.2048	3	$N(\xi)$
	000101	6,5	−2.7039	3	$N(\xi)$
	010111	6,23	6.9087	3	$N(\xi)$

and (ii) one single polynomial parameterizes simultaneously all possible period-k orbits. Algebraic elimination provides as a byproduct an additional polynomial, of degree M_k, which we call $S_k(\sigma)$. When substituted in $P_k(x;\sigma)$, the M_k roots $\{\sigma_\ell\}$ of $S_k(\sigma) = 0$ generate M_k polynomials of degree k, namely $\{P_k(x;\sigma_\ell), \ell = 1, 2, \ldots, M_k\}$. The k roots of these polynomials are the k periodic points of the corresponding cycle. Although we concentrate here on presenting results for a specific situation, it is important to stress that the methodology introduced in Refs. [6,7] to obtain resultants is very general and applies to any dynamical system of algebraic origin [10], i.e. to any dynamical system having equations of motion defined by rational maps, polynomials in particular.

Instead of using Eq. (1), we prefer to change to a new variable $x \equiv ay$ and work with the equivalent but more convenient form

$$x_{t+1} = a - x_t^2 + bx_{t-1}, \quad b \equiv -1. \tag{2}$$

The advantage of this equation is that it generates *monic* minimal polynomials, i.e. polynomials having 1 as the leading coefficient. This implies automatically that all coordinates of periodic points of Eq. (2) are defined in terms of well-studied algebraic quantities known as algebraic *integers* [4,5]. It also means that certain remarkably important algebraic integers, known as *units*, are ruling phase diagrams [11]. The use of algebraic integers enables the trained eye to spot immediately in Table 1 all pairs of σ which are conjugate *quadratic* integers: Those having identical decimal parts. Sums based in Eq. (1) do not show this simplification. Periodic orbit sums σ obtained from Eq. (2) are listed in Table 1. The numerical value of σ is seen to be an integer whenever its algebraic degree is 1. Conjugate pairs of quadratic irrationalities have identical decimal parts and are shown in bold. The orbital sums of Refs. [1,2] are obtained from those in Table 1 by dividing σ by $a = 6$.

Table 2
Expressions $S_k(\sigma)$ defining precisely the approximate sums σ listed in Table 1

k	$S_k(\sigma)$
1	$\sigma^2 + 2\sigma - 6$
2	$\sigma - 2$
3	$\sigma^2 - 2\sigma - 4$
4	$\sigma(\sigma^2 - 24)$
5	$\sigma^6 - 2\sigma^5 - 78\sigma^4 + 240\sigma^3 + 840\sigma^2 - 2496\sigma + 1448$
6	$(\sigma - 2)^2(\sigma^2 + 4\sigma - 24)(\sigma^2 - 6\sigma - 18)(\sigma^3 + 8\sigma^2 - 70\sigma - 228)$

A sufficient condition for the existence of a horseshoe was derived long ago by Devaney and Nitecki [12]. The value $a = 6$ considered previously in Refs. [1,2,13] and studied here is chosen for no particular reason than just simple convenience. From numerical work by Sterling et al. [13] on the Hamiltonian Hénon map it is known that for $a > a_h = 1/(0.41887\ldots)^2 = 5.69931\ldots$ all 2^k periodic points are real. The value $a = 6$ is the integer nearest to a_h. While the integer choice of a reveals relations that might otherwise have been missed, the generic factorizations reported here are representative of what we find for other values of a.

The polynomials $P_k(x)$ and $S_k(\sigma)$ needed to explain the center of mass puzzle are particular cases of more general expressions obtained previously [6–8] which will be recalled only as needed. In Table 2 one finds a summary the key polynomials $S_k(\sigma)$ defining the sum of orbital points for $k \leqslant 6$.

3. Exact equations for orbits and sums

The sums of orbital coordinates for the first four periodic orbits are not difficult to obtain and are not particularly illuminating since they are quite simple. Expressions giving these sums are simply listed in Table 2. Integer sums arise for $k = 2$ and 4 due to the rather simple structure of $S_2(\sigma)$ and $S_4(\sigma)$. The irrationalities α and β underlying orbital points and defining non-diagonal period-4 orbits are $\alpha = \sqrt{6 - 2\sqrt{6}}$ and $\beta = \sqrt{6 + 2\sqrt{6}}$. The first non-trivial case is that for $k = 5$. Since coordinates of period-5 orbits involve 5 numbers and the degree of $S_5(\sigma)$ is 6 (see Table 2), for rational values of a one would expect period-5 orbits to be generically ruled by algebraic numbers of degree $5 \times 6 = 30$, a quite high degree. However, for arbitrary a we find all 30 coordinates of the six period 5 orbits to be simply sextic numbers amalgamated into three remarkably symmetric factors as follows:

$$\mathbb{A}^{(5)}(x) = X(x)Y^2(x)Z^2(x), \tag{3}$$

which for $a = 6$ read

$$X(x) = x^6 - 2x^5 - 14x^4 + 24x^3 + 32x^2 - 16x - 8, \tag{4}$$

$$Y(x) = x^6 - 2x^5 - 16x^4 + 26x^3 + 81x^2 - 84x - 125, \tag{5}$$

$$Z(x) = x^6 + 2x^5 - 16x^4 - 22x^3 + 85x^2 + 60x - 151. \tag{6}$$

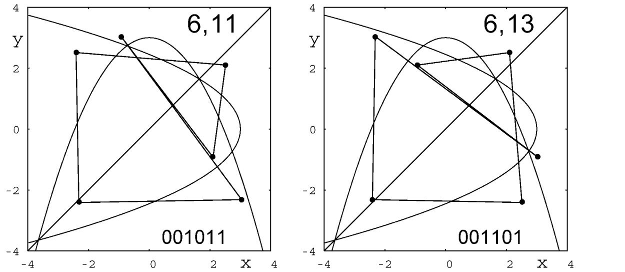

Fig. 1. Non self-symmetric chiral partners of period 6 generated by $C_6(\sigma)$. Under reflection about the symmetry line $y = x$ the 6, 11 cycle maps into its 6, 13 partner, and vice versa.

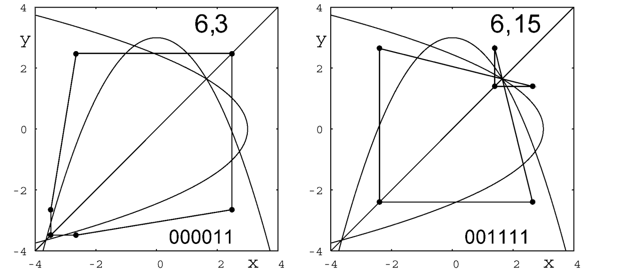

Fig. 2. Self-symmetric pairs of period-6 diagonal cycles defined by $D_6(\sigma)$ in Eq. (9). Each cycle has two points on the diagonal $y = x$ and under reflection about this symmetry line maps into itself.

The Galois group of these three polynomials and of $S_5(\sigma)$ is the full symmetric group of order 6. According to Galois' theory this means that there is no way to express the roots of these polynomials using any *finite* quantity of radicals. In other words, these polynomials are not soluble by radicals. Apart from a factor 2^{12} for $X(x)$, the discriminants of the above trio of sextics coincide, being $\Delta = (2^6)(31)(241)(389)$. Obviously, each member of the irreducible trio generates exactly the same number field. The irrationality χ characterizing the algebraic nature of the six period-5 orbits is that underlying the roots of Eqs. (4)–(6).

A curious fact is that the coordinates of diagonal points of all six 5-cycles conjure to be roots of the same sextic factor: $Z(x)$. Non-diagonal points segregate similarly. This symmetric amalgamation of orbital coordinates reflects the entanglement and conjugate nature of the underlying orbital points and numbers. The factorization in trios $X(x)$, $Y(x)$, $Z(x)$ like in Eq. (3) remains valid for arbitrary choices of a. This fact poses interesting problems: how to construct automorphisms allowing the passage from one orbit to the others and back? Which is the minimum basis field which would allow such automorphisms to exist? Nonlinear interdependencies of a similar kind were already spotted [14–17].

The polynomials encoding all period-6 orbits for arbitrary values of a and b are given in Ref. [7]. For arbitrary values of a one finds that the ninth degree equation $S_6(\sigma)$ breaks into three pieces:

$$S_6(\sigma) = C_6^2(\sigma) D_6(\sigma) N_6(\sigma), \tag{7}$$

where

$$C_6(\sigma) = \sigma - 2, \tag{8}$$

$$D_6(\sigma) = \sigma^2 + 4\sigma - 4a, \tag{9}$$

$$N_6(\sigma) = \sigma^5 + 2\sigma^4 - 4(5a+4)\sigma^3 + 8\sigma^2 a + 4\big(16a^2 + 12a + 9\big)\sigma + 128a^2 - 96a + 72. \tag{10}$$

Factor $C_6(\sigma)$: The sum of the orbital points obtained from the *chiral* factor $C_6(\sigma)$ is always 2, independently of a, and this pair of orbits explains the integers in Table 1. These orbits are labeled 6, 11 and 6, 13 and shown in Fig. 1. Their chiral symmetry is easy to recognize. The parabolas in the background of Fig. 1 and subsequent figures display time-reversal symmetry lines [18,19]. The orbital coordinates of both orbits are roots of the same polynomial

$$O(x) = x^6 - 2x^5 - 14x^4 + 22x^3 + 62x^2 - 60x - 83, \tag{11}$$

that decomposes into two *casus irreducibilis* of cubics over $\mathbb{Q}(\sqrt{3})$, having three real roots despite the intermediate complex radical extension. The basic irrationality underlying orbital coordinates of the chiral orbits is

$$\varphi = \sqrt[3]{61 + 96\sqrt{3} - 3\sqrt{-1599}}. \tag{12}$$

Both cycles 6, 11 and 6, 13 are formed with the *same* the roots of Eq. (11), after adjusting the proper *phase* of the initial conditions.

Factor $D_6(\sigma)$: Self-symmetric orbits having *diagonal* points are obtained from $D_6(\sigma)$, which for $a = 6$ has the roots

$$\sigma_{6,15} = -2 + 2\sqrt{7} \approx 3.2915, \tag{13}$$

$$\sigma_{6,3} = -2 - 2\sqrt{7} \approx -7.2915, \tag{14}$$

with the corresponding orbits being

$$O_{6,15} = \big(x^2 + x - 6 + \sqrt{7}\big)^2 \big(x - \sqrt{7}\big)^2, \tag{15}$$

$$O_{6,3} = \big(x^2 + x - 6 - \sqrt{7}\big)^2 \big(x + \sqrt{7}\big)^2. \tag{16}$$

Each orbit is self-symmetric and has two points on the diagonal as illustrated in Fig. 2.

To conveniently manifest all conjugates among the coordinates of orbital points we label roots as follows: $\theta = \sqrt{7}$, $\bar{\theta} \equiv -\theta$, and

$$u = \frac{1}{2}\Big[-1 + \sqrt{25 - 4\sqrt{7}}\Big] = 1.398, \tag{17}$$

$$\bar{u} = \frac{1}{2}\Big[-1 - \sqrt{25 - 4\sqrt{7}}\Big] = -2.398, \tag{18}$$

$$v = \frac{1}{2}\Big[-1 + \sqrt{25 + 4\sqrt{7}}\Big] = 2.482, \tag{19}$$

$$\bar{v} = \frac{1}{2}\Big[-1 - \sqrt{25 + 4\sqrt{7}}\Big] = -3.482. \tag{20}$$

Then, we introduce "2D orderings" by representing sequences of points

$$(\theta, u) \to (\bar{u}, \theta) \to (\bar{u}, \bar{u}) \to (\theta, \bar{u}) \to \cdots \tag{21}$$

in vertical columns. The periodic ordering obtained for $\sigma_{6,15}$ reads

$$O_{6,15} = \begin{pmatrix} \theta & \bar{u} & \bar{u} & \theta & u & u \\ u & \theta & \bar{u} & \bar{u} & \theta & u \end{pmatrix} \tag{22}$$

while the *conjugate* ordering for $\sigma_{6,3}$ is

$$O_{6,3} = \begin{pmatrix} \bar\theta & \bar v & \bar v & \bar\theta & v & v \\ v & \bar\theta & \bar v & \bar v & \bar\theta & v \end{pmatrix}. \tag{23}$$

As is clear, the exact analytical representations of the coordinates manifest the conjugacy explicitly in a way not possible using representations based on the approximate numbers given on the right-hand side of Eqs. (17)–(20), which correspond to "projections" of the algebraic coordinates onto the real line. The conjugate nature of the algebraic numbers defining coordinates also plainly shows why quartic coordinates add to produce quadratic sums. Such conjugacies are generic for higher periods.

Factor $N_6(\sigma)$: Self-symmetric orbits having only *non-diagonal* points are obtained from $N_6(\sigma)$, which for $a = 6$ decomposes into the product

$$N_6(\sigma) = (\sigma^2 - 6\sigma - 18)(\sigma^3 + 8\sigma^2 - 70\sigma - 228). \tag{24}$$

The quadratic factor gives $\sigma_{6,31} = 3 + 3\sqrt{3} \approx 8.1961$ and $\sigma_{6,7} = 3 - 3\sqrt{3} \approx -2.1961$. The orbital equation for $\sigma_{6,31}$ is

$$O_{6,31} = (x^2 + x - x\sqrt{3} - 2)(x^2 - 2x - x\sqrt{3} - 2 + 3\sqrt{3})^2 \tag{25}$$

involving four distinct coordinates

$$\theta = \frac{1}{2}\Big[-1 + \sqrt{3} + \sqrt{12 - 2\sqrt{3}}\,\Big] = 1.826,$$
$$\bar\theta = \frac{1}{2}\Big[-1 + \sqrt{3} - \sqrt{12 - 2\sqrt{3}}\,\Big] = -1.094,$$
$$u = \frac{1}{2}\Big[2 + \sqrt{3} + \sqrt{15 - 8\sqrt{3}}\,\Big] = 2.400,$$
$$\bar u = \frac{1}{2}\Big[2 + \sqrt{3} - \sqrt{15 - 8\sqrt{3}}\,\Big] = 1.331$$

and 2D ordering

$$O_{6,31} = \begin{pmatrix} \theta & \bar u & u & \bar\theta & u & \bar u \\ \bar u & \theta & \bar u & u & \bar\theta & u \end{pmatrix}. \tag{26}$$

Similarly, the orbit conjugated to $O_{6,31}$ is

$$O_{6,7} = (x^2 + x + x\sqrt{3} - 2)(x^2 - 2x + x\sqrt{3} - 2 - 3\sqrt{3})^2, \tag{27}$$

involving another four distinct coordinates

$$v = \frac{1}{2}\Big[-1 - \sqrt{3} + \sqrt{12 + 2\sqrt{3}}\,\Big] = 0.600,$$
$$\bar v = \frac{1}{2}\Big[-1 - \sqrt{3} - \sqrt{12 + 2\sqrt{3}}\,\Big] = -3.332,$$
$$w = \frac{1}{2}\Big[2 - \sqrt{3} + \sqrt{15 + 8\sqrt{3}}\,\Big] = 2.819,$$
$$\bar w = \frac{1}{2}\Big[2 - \sqrt{3} - \sqrt{15 + 8\sqrt{3}}\,\Big] = -2.551$$

and 2D ordering

$$O_{6,7} = \begin{pmatrix} v & w & \bar w & \bar v & \bar w & w \\ w & v & w & \bar w & \bar v & \bar w \end{pmatrix}. \tag{28}$$

The orbits $O_{6,7}$ and $O_{6,31}$ are shown in Fig. 3.

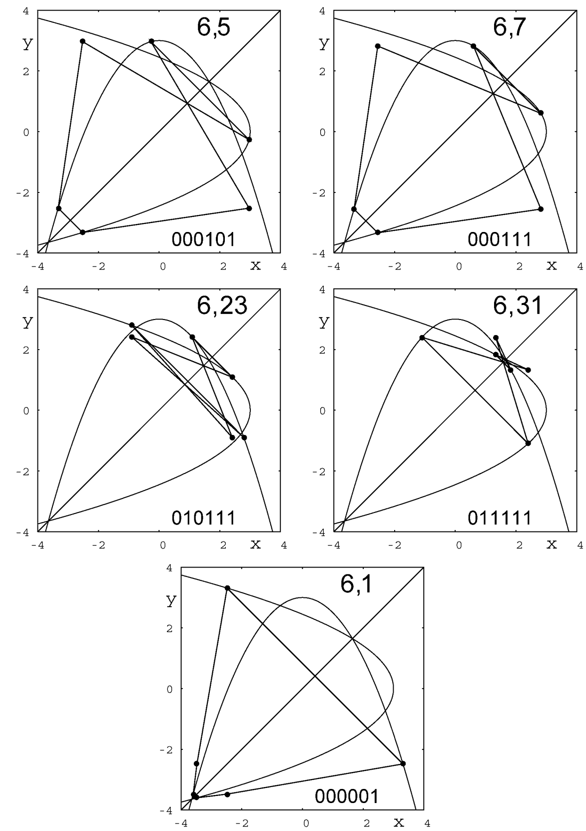

Fig. 3. The five self-symmetric period-6 non-diagonal cycles defined by $N_6(\sigma)$ in Eq. (24). The cycles 6, 7 and 6, 31 arise from the quadratic factor in Eq. (24) while the other three cycles arise from the cubic factor.

The roots of the cubic in Eq. (24) are $\sigma_{6,1} \simeq -12.2048$, $\sigma_{6,5} \simeq -2.7039$ and $\sigma_{6,23} \simeq 6.9087$ and the corresponding orbits are also shown in Fig. 3.

The amalgamations of the pairs of orbits arising from the quadratic factors are

$$\mathbb{A}^{(6)}_{3,15} = (x^4 + 2x^3 - 11x^2 - 12x + 29)^2(x^2 - 7)^2, \tag{29}$$

$$\mathbb{A}^{(6)}_{7,31} = (x^4 - 4x^3 - 3x^2 + 26x - 23)^2 \times (x^4 + 2x^3 - 6x^2 - 4x + 4) \tag{30}$$

while the amalgamation of the trio of orbits due to the cubic in Eq. (24) is

$$\mathbb{A}^{(6)}_{1,5,23} = h_1(x)h_2^2(x), \tag{31}$$

where

$$h_1(x) = x^6 - 22x^4 + 8x^3 + 124x^2 - 88x - 32, \tag{32}$$

$$h_2(x) = x^6 + 4x^5 - 12x^4 - 58x^3 + 12x^2 + 202x + 139. \tag{33}$$

The trio of orbits defined by the roots of $h_1(x)$ and $h_2(x)$ factorize in the sextic field $\mathbb{Q}(\xi)$ (indicated in Table 1) and defined by

$$\xi = (46 + 102\sqrt{-1977})^{1/3}. \tag{34}$$

Similarly to what happens in the *casus irreducibilis* of cubics, real roots of the sextics above are expressed by radicals of complex numbers, not by real radicals. In other words, such roots are contained in a radical extension of the real base field, although not in a real radical extension of the base field. Of course, all quantities derived from ξ are real, as may be easily verified.

The peculiar amalgamations above are very different from similar ones found for the purely chaotic case involving "inheritance" [14] for $a = 2$, the parameter found useful to emulate logic gates, to encode numbers, and to perform specific arithmetic operations on those numbers such as addition and multiplication [20].

4. Conclusions

The center of mass puzzle arises from factorizations of the polynomials $S_k(\sigma)$ in Table 2. For rational values of a, the presence of factors linear in σ in $S_k(\sigma)$ implies necessarily the emergence of rational values of σ, integers in particular. Similarly, quadratic factors in $S_k(\sigma)$ imply quadratic values σ. Analogous reductions are obviously present for factors of all higher orders but the underlying symmetries and "simplicity" in orbital coordinates will become evident only when computing them exactly. There is no hope to recognize symmetries from high-order algebraic quantities ruling the dynamics looking at the typical "projections" on the real line familiar from approximate numerical work. It is clear that the much simpler algebraic nature of σ (compared to that of the orbital coordinates defining it) arises from cancellations occurring when adding sets containing conjugate algebraic numbers (not necessarily all conjugates). Thus, the nature of σ is always simpler than the nature of the individual orbital coordinates.

An useful byproduct of our computations are the amalgamations given above. They define orbital points in phase space precisely and are valuable test-ground for developing algorithms aiming at the systematic search of periodic orbits. The roots of the polynomials defining amalgamations provide with *arbitrary precision* the full set of orbits, something that is not trivial to find numerically when the period increases. The intricacies of efficiently detecting periodic orbits in chaotic systems have been recently reviewed by Crofts and Davidchack [21]. Among other things, they discuss the computation of unstable periodic orbits for three coupled Hénon maps. As they explain, with increasing period basin sizes become so small that placing initial seeds in them for search algorithms to start becomes practically impossible. We already have exact orbits, amalgamations and $S_k(\sigma)$ up to period $k = 11$ [8].

In a separate work [8] we show that the classes C, D, and N discussed briefly here are in fact generic and imply "clusters of orbits" dictated by the algebraic irrationality underlying each class. As may be easily recognised from Table 1, the classes imply reshufflings and mandatory interdependencies between the familiar binary labels which thus far have been always considered as representing orbits totally independent from each other. To conclude, we would like to mention that the enticing question concerning possible interconnections between the peculiar factorizations reported here and the reversibility in generic Hamiltonian systems [18,19] is addressed in a separate work dealing with exact orbits of considerably higher period.

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