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Nonlinear dependencies between sets of periodic orbits

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Abstract. – We report the existence of a phenomenon of *inheritance* among periodic trajectories of the quadratic map. Within the set $\mathbf{S}_k$ of all possible trajectories of period k we find "mother" trajectories from which several "daughters" may be derived by simple polynomial transformations. For example, from the six orbital points z_i of a period-six mother we get three additional period-six daughters as the zeros of the six cubics $x^3 - 3x - z_i = 0$. Daughters might have daughters. This stratification shows that periodic orbits are not necessarily independent of each other. This fact could be of importance for decomposing certain sums involving sets of periodic trajectories, particularly for trace formulas underlying semiclassical interpretations of spectra in atomic physics.

The infinite set of periodic orbits embedded in chaotic attractors is fundamental for an understanding of, for example, chaotic experimental time series and of chaotic dynamics in general. The knowledge of the structure of periodic orbits [1] underlying physical attractors can be used to determine basic ergodic properties such as dimension, Lyapunov exponents, and topological entropy, properties that are basic tools to characterize physical processes. Periodic orbits play a central role in the interpretation of quantum-mechanical spectra of systems whose classical counterpart exhibit chaotic behavior [2, 3]. So, the investigation of periodic orbits is a major issue nowadays [4].

An open problem in this field is the determination of the total number of isoperiodic orbits, *i.e.* orbits sharing the same period, for generic dynamical systems. In some cases this number is known to grow super-exponentially fast [5] but so far there is no general prescription for estimating the abundant proliferation of periodic orbits for generic systems. Interesting questions still awaiting solution exist also *within* the set $\mathbf{S}_k$ of all orbits of period k of a given dynamical system, and the purpose of this paper is to report an unexpected and curious answer to the following simple question: are the orbits in $\mathbf{S}_k$ always independent of each other? Or,

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equivalently, is it possible to find explicit examples of systems having periodic orbits which although looking totally independent in numerical simulations are nevertheless interconnected?

This paper reports a novel phenomenon of "period inheritance" found in sets of isoperiodic orbits. Inheritance is characterized by the existence of copious sets of orbits within orbits: under suitable conditions, the knowledge of a single periodic orbit may be used to infer and explicitly obtain many additional ones. Underlying inheritance there is an intricate set of transformations among the numerical values of the points forming the various periodic orbits, more precisely, an intricate set of automorphisms interconnecting pieces into which the equations of motion can be decomposed.

Interdependencies among periodic orbits are hard to recognize in finite precision numerical simulations before explicitly establishing the exact mathematical expression of the transformation. Our objective here is to illustrate their existence by presenting several explicit examples. Inheritances could be of practical interest, specially for applications in atomic physics, implying the possibility of rearranging orbit-dependent contributions in semiclassical sums needed for calculations of energy spectra and density of states with, *e.g.*, Gutzwiller's trace formula [2,3].

Inheritance can be more easily understood in the context of a specific system. Therefore, we illustrate it for the most familiar example of nonlinear dissipative system, namely for the quadratic map [6]

$$x_{t+1} = a - x_t^2, \qquad t = 0, 1, 2, \cdots, \qquad x_0 = x, \tag{1}$$

with $a = 2$. This equation with the same parameter has been recently shown by Sinha and Ditto [7] to be the heart of a lattice of coupled chaotic maps able to perform simple computations, to emulate logic gates used in the construction of computers, to encode numbers and to perform a whole variety of related arithmetic operations. It is important to emphasize, however, that inheritance is by no means restricted to eq. (1). It also exists for a (quartic) model of a ring cavity used in the construction of a certain type of lasers [8, 9], for the paradigmatic $(x, y) \mapsto (a - x^2 + by, x)$ Hénon map [10], including the Hamiltonian $b = \pm 1$ limits, and for Hamiltonian (conservative) systems constructed *ad hoc* to investigate inheritance. Here we concentrate on the simplest example.

When iterated, eq. (1) produces an infinite family of polynomials, which we denote by $\tilde{p}_k(x)$, for $k = 1, 2, \ldots$, k being the number of iterates needed to generate $\tilde{p}_k(x)$. All roots of this infinite set of polynomials are real [11]. To exhibit the properties of the $\tilde{p}_k(x)$ more easily, we write down the first few *in extenso*:

$$\tilde{p}_1(x) = x^2 + x - 2 = (x + 2)\,(x - 1), \tag{2}$$

$$\tilde{p}_2(x) = \tilde{p}_1(x)(x^2 - x - 1), \tag{3}$$

$$\tilde{p}_3(x) = \tilde{p}_1(x)(x^3 - x^2 - 2\,x + 1)(x^3 - 3\,x - 1), \tag{4}$$

$$\begin{aligned}\tilde{p}_4(x) = {} & \tilde{p}_1(x)(x^2 - x - 1)(x^4 + x^3 - 4\,x^2 - 4\,x + 1) \times \\ & \times (x^8 - x^7 - 7\,x^6 + 6\,x^5 + 15\,x^4 - 10\,x^3 - 10\,x^2 + 4\,x + 1).\end{aligned} \tag{5}$$

As might be seen, the zeros of every $\tilde{p}_k(x)$ contain all possible solutions of period k, including trivial ones (the repeated factors), which are not proper orbits of period k because they already appeared earlier as zeros of equations $\tilde{p}_\ell(x)$ with ℓ a divisor of k. For example, the trivial zeros of $\tilde{p}_2(x) = 0$ are $x = 1$ and $x = -2$, the fixed point solutions of $\tilde{p}_1(x) = 0$. The zeros of $x^2 - x - 1 = 0$ correspond to a genuine orbit of period two. The fixed points, $\tilde{p}_1(x)$, appear as trivial factors in every $\tilde{p}_k(x)$, for $k > 1$.

All trivial periodicities may be discarded by polynomial division. Removal of trivial periodicities generates a new family of polynomials, denoted by $p_k(x)$, containing by construction *all* genuine motions of period k and no other periods:

$$p_1(x) = (x+2)(x-1), \tag{6}$$
$$p_2(x) = x^2 - x - 1, \tag{7}$$
$$p_3(x) = (x^3 - x^2 - 2\,x + 1)(x^3 - 3\,x - 1), \tag{8}$$
$$p_4(x) = (x^4 + x^3 - 4\,x^2 - 4\,x + 1) \times \\ \times (x^8 - x^7 - 7\,x^6 + 6\,x^5 + 15\,x^4 - 10\,x^3 - 10\,x^2 + 4\,x + 1). \tag{9}$$

The degree of every $p_k(x)$ is a multiple of k. Sometimes $p_k(x)$ decomposes quite naturally into factors whose individual degrees are also multiples of k, the multiplicity indicating the quantity of k-periodic orbits defined by their zeros. So, there are two distinct orbits of period 3, and three orbits of period 4, two of them defined by the zeros of the octic.

The families $\tilde{p}_k(x)$ and $p_k(x)$ are very simply related:

$$\begin{array}{ll} \tilde{p}_1(x) = p_1(x), & \tilde{p}_2(x) = p_1(x)\,p_2(x), \\ \tilde{p}_3(x) = p_1(x)\,p_3(x), & \tilde{p}_4(x) = p_1(x)\,p_2(x)\,p_4(x), \\ \tilde{p}_5(x) = p_1(x)\,p_5(x), & \tilde{p}_6(x) = p_1(x)\,p_2(x)\,p_3(x)\,p_6(x), \\ \tilde{p}_7(x) = p_1(x)\,p_7(x), & \tilde{p}_8(x) = p_1(x)\,p_2(x)\,p_4(x)\,p_8(x), \end{array}$$

where we included a few additional decompositions that are easy to obtain but too long to display here explicitly. From this last set of equations one easily recognizes an interesting fact: The process of removing trivial periodicities resolved the several $\tilde{p}_k(x)$ into a natural "substructure" $p_k(x)$ which can be labeled with the same numerical factors found by decomposing the period k into prime integers. For example, $\tilde{p}_6(x)$ is divisible by $\tilde{p}_1(x)$, by $\tilde{p}_2(x)$, by $\tilde{p}_3(x)$ and by $\tilde{p}_6(x)$ itself. If k is prime, then $\tilde{p}_k(x)$ is also prime in the sense that it is divisible only by $\tilde{p}_1(x)$ and by $\tilde{p}_k(x)$ itself. In addition, for all prime periods k one finds that $\tilde{p}_k(x)$ may always be decomposed into a product of two factors, $\tilde{p}_k(x) = p_1(x)\,p_k(x)$, where the trivial "identity" factor $p_1(x) = (x+2)(x-1)$ corresponds to the well-known fixed points of the dynamics.

In the above framework, the problem of determining periodic trajectories is equivalent to the problem of characterizing the internal number-theoretical structure of the irreducible factors composing $p_k(x)$, *i.e.* equivalent to determining the towers of numbers [12] in $p_k(x)$. This one-to-one correspondence remains valid for other choices of parameter in eq. (1) and for other one-dimensional algebraic dynamical systems. It also remains valid for multidimensional systems after suitable elimination of variables. Once again, an understanding of the arithmetical properties of the equations of motion is found to be a prerequisite to understand their dynamics [13].

Now we illustrate inheritance for period six orbits by considering the zeros of $p_6(x)$ explicitly. A simple computation shows that $p_6(x)$ consists of four factors:

$$p_6(x) = \Phi_6^{(1)}(x)\,\Phi_6^{(2)}(x)\,\Phi_{18}(x)\,\Phi_{24}(x), \tag{10}$$

where

$$\Phi_6^{(1)}(x) = x^6 - x^5 - 5x^4 + 4x^3 + 6x^2 - 3x - 1, \tag{11}$$
$$\Phi_6^{(2)}(x) = x^6 + x^5 - 6x^4 - 6x^3 + 8x^2 + 8x + 1, \tag{12}$$
$$\Phi_{18}(x) = x^{18} - 18x^{16} + x^{15} + 135x^{14} - 15x^{13} - 546x^{12} + 90x^{11} + \\ +1287x^{10} - 276x^9 - 1782x^8 + 459x^7 + 1385x^6 - 405x^5 - 534x^4 + \\ +170x^3 + 72x^2 - 24x + 1, \tag{13}$$

Table I. – *Approximate numerical values defining the* 3 *period-six trajectories* $\tau_i^{(j)}$, $j = 1, 2, 3$, *generated by the roots* z_i *of* $\Phi_6^{(2)}(x) = 0$, *obtained by solving the cubic* $x^3 - 3x - z_i = 0$, *eq.* (18).

i	z_i	$\tau_i^{(1)}$	$\tau_i^{(2)}$	$\tau_i^{(3)}$
1	−1.911145	−1.990061	0.822574	1.167487
2	−1.652477	−1.960344	1.323371	0.636973
3	−0.730682	−1.842952	0.248687	1.594265
4	−0.149460	0.049861	−1.756443	1.706581
5	1.466103	−1.396473	1.938154	−0.541680
6	1.977661	1.997513	−1.085092	−0.912421

$$\Phi_{24}(x) = x^{24} + x^{23} - 24x^{22} - 23x^{21} + \ldots + 101x^3 - 180x^2 + 12x + 1. \tag{14}$$

Equation (14) will not be needed here. It may be generated with computer programs capable of performing algebraic manipulations.

The four factors in eq. (10) are the *minimum polynomials* [12] which fix the arithmetic properties characterizing all motions of period six. From their irreducibility we learn that there are three different classes of period six orbits: From the nine possible orbits there are i) two (different) orbits defined by sextics (*i.e.* over number fields of degree 6; see [12]), ii) three over a field of degree 18, and iii) four over a field of degree 24. This is a precious information for the analysis of the orbital substructuring but we do not go into this here.

All polynomials generated by iterating equations of motion are *Abelian equations* [14] by construction. This means that their zeros might be expressed with a finite number of radicals. In the present case their computation is greatly simplified because the four factors in eq. (10) contain quadratic subfields, *i.e.* each one is a product of two factors, one with coefficients of the form $\alpha_j + \beta_j\sqrt{d}$, α, β, d rationals, the other with conjugate factors $\alpha_j - \beta_j\sqrt{d}$. So, $\Phi_{18}(x)$ contains the subfield $\mathbb{Q}(\sqrt{21}\,)$ while $\Phi_{24}(x)$ may be decomposed in three different ways: over $\mathbb{Q}(\sqrt{5}\,)$, over $\mathbb{Q}(\sqrt{13}\,)$ and over $\mathbb{Q}(\sqrt{65}\,)$. The cubic factors of $\Phi_6^{(1)}(x)$ and $\Phi_6^{(2)}(x)$ are, respectively,

$$\varphi_1(x) = x^3 - \frac{1+\sqrt{13}}{2}x^2 - x + \frac{3+\sqrt{13}}{2}, \tag{15}$$

$$\varphi_2(x) = x^3 + \frac{1-\sqrt{21}}{2}x^2 - \frac{1+\sqrt{21}}{2}x + \frac{5+\sqrt{21}}{2}. \tag{16}$$

Their conjugates $\overline{\varphi}_1(x)$ and $\overline{\varphi}_2(x)$ are obtained from the above ones by changing the sign in front of all square roots. From these cubics, the problem of finding exact analytical expressions for all orbital points of $\Phi_6^{(1)}(x)$ and $\Phi_6^{(2)}$ is simple to solve. The first column of table I shows approximate numerical values for z_i, the six zeros of $\Phi_6^{(2)}(x)$.

From the analytical expressions for the zeros it is not difficult to see that $\Phi_{18}(x)$ inherits its three orbits from the orbit defined by $\Phi_6^{(2)}(x)$. This is guaranteed by the identity

$$\Phi_{18}(x) = \Phi_6^{(2)}(x^3 - 3x). \tag{17}$$

This identity means that once the zeros z_i of the $\Phi_6^{(2)}(x) = 0$ mother are obtained, three different daughters follow from the zeros of the six cubics

$$x^3 - 3x - z_i = 0. \tag{18}$$

This solves $\Phi_{18}(x)$ analytically. Approximate values for these zeros are shown in table I.

The transformation $x^3 - 3x$ plays an even more interesting role when considered within the set of the 56 orbits of period nine, defined by the following factors:

$$p_9(x) = \Psi_9^{(1)}(x)\,\Psi_9^{(2)}(x)\,\Psi_{18}(x)\,\Psi_{36}(x)\,\Psi_{54}(x)\,\Psi_{162}(x)\,\Psi_{216}(x). \tag{19}$$

As for eq. (10), subindices indicate the degree of each polynomial. Once again, there are two different factors of lowest degree, 9, their specific naming being immaterial here.

Period nine displays the first instance of "double inheritance", shown in the second identity below:

$$\Psi_{54}(x) = \Psi_{18}(x^3 - 3x), \tag{20}$$

$$\Psi_{162}(x) = \Psi_{18}(x^9 - 9x^7 + 27x^5 - 30x^3 + 9x) = \Psi_{18}(y^3 - 3y), \tag{21}$$

where $y = x^3 - 3x$ and

$$\begin{aligned}\Psi_{18}(x) = x^{18} &+ x^{17} - 18\,x^{16} - 18\,x^{15} + 134\,x^{14} + 134\,x^{13} - 531\,x^{12} - 531\,x^{11} + \\ &+1198x^{10} + 1198\,x^9 - 1519\,x^8 - 1519\,x^7 + 989\,x^6 + 989\,x^5 - \\ &-265\,x^4 - 265\,x^3 + 20\,x^2 + 20\,x + 1.\end{aligned} \tag{22}$$

Equations (20) and (21) show explicitly an extremely intricate and beautiful "bifurcation" symmetry interconnecting orbits of period nine due to unsuspected links (automorphisms) between specific number-fields which slave the dynamics. From $\Phi_6^{(2)}(x)$ and $\Psi_{18}(x)$ we recognize a trend that remains valid for other transformations: the polynomials generating upwardly heritable orbits define a new family of reciprocal-looking polynomials, since they also contain pairs of identical coefficients, simultaneously reducing their freedom but increasing their symmetry.

As a last example we present another inheritance-generating transformation, one which is considerably more elaborate than those discussed so far. First there appear interconnecting orbits of period ten, defined unambiguously by minimum polynomials of the following degrees:

$$p_{10}(x) = \Theta_{10}(x)\,\Theta_{20}(x)\,\Theta_{30}(x)\,\Theta_{80}(x)\,\Theta_{150}(x)\,\Theta_{300}(x)\,\Theta_{400}(x). \tag{23}$$

Altogether, there are 99 orbits of period 10. Among them inheritance appears through the quintic,

$$\Theta_{400}(x) = \Theta_{80}(x^5 - 5x^3 + 5x), \tag{24}$$

proving the existence a whole new class of nonlinear transformations. Notice that the map $f(x) = 2 - x^2$ *and* the transformations $u(x) = x^3 - 3x$ and $v(x) = x^5 - 5x^3 + 5x$ commute under functional composition: $f(u(x)) = u(f(x))$, $f(v(x)) = v(f(x))$. In addition, $u(v(x)) = v(u(x))$. A full discussion of inheritance-generating conjugacy relations will be presented elsewhere.

We conclude with a few remarks about the computations needed to obtain the results above.

The explicit determination of the aforementioned transformations required algebraic computations with quite large polynomials involving huge numerical coefficients and related quantities. For example, the discriminant of $\Theta_{400}(x)$, in eq. (24), is $5^{700}41^{390}$, a number of 1119 digits that requires a considerable amount of time to compute. Similarly, classifying all 186 motions of period 11 requires dealing with highly peculiar and symmetrical numbers like, for example, $3^{341}683^{681}$, containing 2093 digits. The size of such numbers greatly increases as the period increases.

All interdependencies reported here are guaranteed to exist because it was possible to establish their links analytically. However, it seems important to point out that in most practical applications periodic orbits are determined numerically with finite precision. Finite precision hinders the recognition of orbit interdependencies and only exact arithmetical work seems capable of revealing their existence unambiguously.

Interdependencies among periodic orbits have consequences for the theoretical description of dynamical systems. An interesting one is their implication for the understanding of the structuring of invariant manifolds emanating from periodic orbits living in basin-boundaries and ruling the dynamics [15]. Since there are interconnections among sets of orbits, it seems also reasonable to expect interconnections between corresponding manifolds which, in principle, would greatly reduce their geometrical structure. Thus, in some cases, manifolds that could appear to be at first rather complicate might be, in fact, much simpler than anticipated. The bad news is that, so far, we have been only able to find inheritance among orbits of relatively high periods, where there is little hope anyway for describing the complicated interlacing of manifolds. This is certainly the case for all typical low-degree paradigmatic models used nowadays. However, it does not rule out the possibility of finding dynamical systems involving just a few monomials of high degrees and capable of displaying inheritance already at low periods and, as a consequence, an extraordinarily organized structure of manifolds.

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