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Discovering parameter values by measuring self-similar structures in the phase-space of dissipative systems with constant Jacobian

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Abstract. – This paper shows that dissipative dynamical systems with constant Jacobian allow one to discover the numerical values of physical parameters under which the system is operating. This is done by performing measurements on self-similar (fractal) structures of the phase-space. Parameter recovery is illustrated explicitly for the Ikeda laser ring-cavity map and the Hénon map. The first model involves transcendental equations of motion that can be solved only numerically. Analytical results are obtained for the second model. In both cases the macroscopic dissipation rate of the dynamical system is recovered from the speed at which fractal "fingers" making up basins of attraction accumulate towards basin boundaries.

Many interesting results in physics during the last hundred years have been obtained from the phase-space. For instance, the phase-space played a decisive role for the idea of quantization, from the old Bohr-Sommerfeld rule up to the Einstein-Brillouin-Keller (EBK) quantization [1], when integrals of the action in phase-space were found to be related to Planck's constant h. To this day, such relation has had many important implications and applications. The phase-space allows one to recognize that several different quantum systems may share a common classical limit [2]. More recently, with the realization of the ubiquity of deterministic chaos at a classical level and with the search for the implications that chaotic classical dynamics might have at a quantum level [3], the investigation of phase-space has acquired a renewed importance. For instance, phase-space was shown to possess new and unanticipated properties like, for example, to have intrinsic self-similar "fractal" structures [4]. A different sort of intricate structures in phase-pace attracting much attention nowadays are

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the so-called *riddled* basins [5]. The sensitivity to initial conditions in phase-spaces with riddled basins is so extreme that no matter how small a volume is chosen, it will always contain initial conditions leading to different final states. This implies unpredictability at all scales of resolution. Another very interesting aspect of the phase-space was found recently by Beck [6] while studying the dynamics of a kicked charged particle moving in a double-well potential in a time-dependent magnetic field. He showed that certain classical particle dynamics possess the complex logistic map as its stroboscopic mapping, thereby attaching a direct physical meaning to the corresponding Julia and Mandelbrot sets [7] which, so far, were believed to be just abstract mathematical objects. At a different level, it is also revealing that "The Weyl representation places quantum mechanics in the phase-space" [8].

The phase-space typical of dissipative systems is known to be subdivided, over wide parameter ranges, into several basins of attraction, one for each stable motion (attractor) supported by the system. The invariant boundaries separating such basins may be either smooth or fractal [4] and involve a hierarchical self-similar alternation of components which typically accumulate on singularities like, *e.g.*, unstable fixed points or periodic orbits [9].

The purpose of this paper is to report a generic property of the self-similar structures which compose the invariant basin boundaries of dissipative systems with constant Jacobian. The parameter space of such systems contains certain *eigenvalue paths*, namely parameter paths along which it is possible to discover the numerical value of parameters ruling the dynamics by performing a standard stability analysis around fixed points [10]. Eigenvalue paths are obtained from the Jacobian matrix of the system by relating two independent quantities obtained from it that control the dynamics. The two quantities of interest are the eigenvalue of largest magnitude and the determinant J of the Jacobian matrix, *i.e.* the Jacobian of the map. One finds eigenvalue paths by interconnecting these two quantities.

Parameter recovery is illustrated now for two familiar models: the Ikeda laser ring-cavity map [11, 12] and the Hénon map [9, 13]. The Ikeda laser ring-cavity map has transcendental equations of motions and parameters for it must be recovered numerically. In spite of this technical nuisance, the laser ring-cavity model is of great interest because it may be studied experimentally in the laboratory [11, 12]. The Hénon map is a paradigmatic model whose wide-ranging *analytical* results are easy to obtain. The possibility of recovering parameters from the geometrical structuring of the phase-space of chaotic systems is a useful theoretical twist because eigenvalue paths provide privileged parameter loci along which it is particularly fruitful to perform a plethora of numerical experiments, to study scalings, to study specific arithmetical properties underlying bifurcation cascades, etc.

We start with the simpler case, deriving analytically the eigenvalue path for the Hénon map $(x, y) \mapsto (a - x^2 + by, x)$, where x, y denote the usual variables and a, b the parameters. The dissipation rate is given by the Jacobian $J = -b$. The volume contraction of dissipative systems can be evaluated using the Lie derivative along the dynamical vector fields [14]. The pair of eigenvalues corresponding to the Hénon map are $\lambda_\pm = -x_u \pm (x_u^2 + b)^{1/2}$, where (x_u, x_u) is the location of the unstable fixed point: $2\,x_u = b - 1 - \{(b-1)^2 + 4a\}^{1/2}$. Substituting x_u into the expression for the eigenvalues one finds an expression for λ_+ involving only a and b. Then, from the relation $\lambda_+ = 1/J = 1/(-b)$ connecting contraction and dissipation, after some simple algebra we arrive at the following expression for the eigenvalue path:

$$W(a, b) \equiv 4ab^2 - (b^2 + b + 1)(b^2 + 3b + 1)(b - 1)^2 = 0. \tag{1}$$

Figure 1 shows the parameter loci U and L obtained by solving the equation $W(a, b) = 0$. Noteworthy along L is the point p where *three different curves meet*: in addition to the eigenvalue path, p also belongs to the $1 \to 2$ bifurcation boundary *and* to the saddle-node

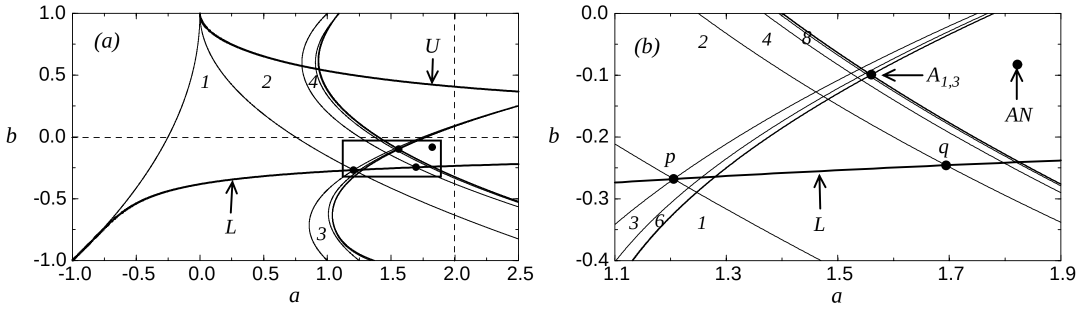

Fig. 1 – (a) The "U"pper and "L"ower portions of the parameter path obtained by solving eq. (1), $W(a,b)=0$, superimposed over salient features in parameter space, particularly over the lines delimiting the first few stability domains for the 1×2^n and 3×2^m bifurcation cascades; (b) magnification of the box in (a) showing the location of p. The phase-space at p is shown in fig. 2. AN indicates the location of the anti-nose [12] characterized by several discontinuities; $A_{1,3}$ is the point of double accumulation, *i.e.* the point where both bifurcation cascades, 1×2^n and 3×2^m, accumulate simultaneously [16].

bifurcation line along which orbits of period 3 are born. The point p has several interesting arithmetical properties [15,16], *e.g.*, *commensurate* coordinates: $p=(a,b)=\big(-9b^*/2,\ b^*\big)$, where $b^*=-2+\sqrt{3}\simeq -0.267949192$. This number is an algebraic *unit* [17]. The phase-space corresponding to the parameters defining p has a self-similar structure and we now illustrate how to recover the dissipation rate from this structure.

Figure 2 shows portions of the phase-space corresponding to the point p in parameter space. The phase-space is given by three bitmaps, each one computed over a rectangular mesh of $2400\times 1200 = 2.88\times 10^6$ initial conditions. The phase-space is seen to be subdivided into three different basins: the gray shading corresponding to the trivial attractor at infinity, the black basin of a period-3 orbit (located at the vertices of the triangle in fig. 2a), and the white basin of the *stable* fixed point indicated by "s". Figure 2 also shows the *unstable* fixed point $u=(x_u,x_u)$, $x_u=-(9-3\sqrt{3})/2\simeq -1.901923$. Since u lies exactly on the basin boundary, it is a rather privileged reference point.

Figure 2b shows a magnification of the rectangle containing u. Figure 2c is a magnification of a similar rectangle drawn around u in fig. 2b which cannot be seen in this figure because its size is much smaller than the actual size of the dot representing u. Figure 2b contains an auxiliary horizontal line passing through u and intersecting the infinite sequence of vertical doublets of black stripes, which are the basin of attraction of the period-3 orbit. As indicated in the figure, each doublet might be characterized by four coordinates x_i labeling the intervals containing the black stripes. To simplify the discussion, we call intervals like $[x_4,x_1]$ a "finger", labeling them consecutively $f=1,2,\ldots$ in the direction towards the accumulation point on the basin boundary, as indicated schematically by the numbers inside circles near some of them.

Figures 2b and c show the first few of an infinite quantity of such self-similar fingers which accumulate towards the basin boundary at u. The regularity of the accumulation process motivates us to measure two characteristic quantities: i) $v_{\rm a}$, the "accumulation speed", *i.e.* the rate at which fingers accumulate towards the basin boundary of the attractor at infinity, and ii) $r_{\rm c}$, the "compression rate", *i.e.* the ratio with which the width of the pair of stripes composing each finger gets compressed as fingers move closer and closer to the basin of the attractor at infinity. We measure these quantities along the horizontal reference line containing u.

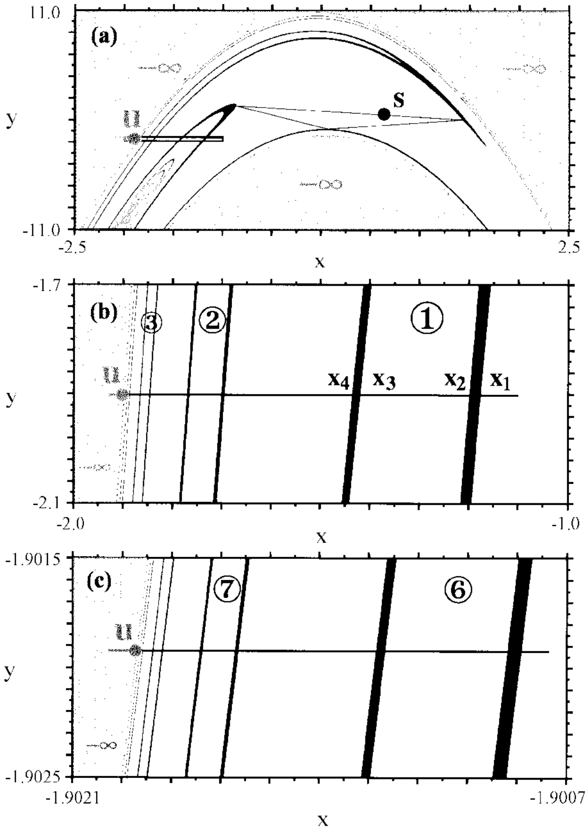

Fig. 2

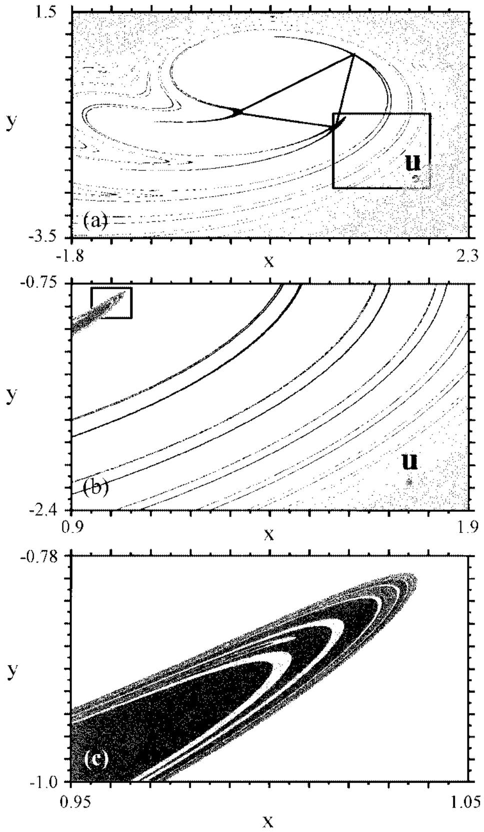

Fig. 3

Fig. 2 – Phase-space displaying self-similar "fingers" $f = 1, 2, \ldots$, labeled by the numbers inside circles. Each finger contains a doublet of black stripes delimited by four numbers $x_\ell^{(f)}$, $\ell = 1, 2, 3, 4$ along a line through the unstable fixed point u, as indicated. (b) and (c) show successive magnifications of the rectangle in (a).

Fig. 3 – Successive magnifications showing a double structure of self-similar fingers for the laser ring-cavity map [13] for $\alpha = 0.84753$ and $\beta = 0.83$. The black period-3 fingers "contain" a substructure of period-36 fingers (shown in gray, in (c)). The period-3 orbit is located at the vertices of the triangle in (a) and oscillates clockwise. The black dot indicates the unstable fixed point u on the boundary.

So, with the boundary points $x_i^{(f)}$ we define two "instantaneous quantities" as follows:

$$v_{\mathrm{a}}^{(f)} = \frac{x_1^{(f+2)} - x_1^{(f+1)}}{x_1^{(f+1)} - x_1^{(f)}}, \qquad r_{\mathrm{c}}^{(f)} = \frac{x_1^{(f)} - x_2^{(f)}}{x_3^{(f)} - x_4^{(f)}}, \tag{2}$$

together with their corresponding asymptotic limits

$$v_{\mathrm{a}} = \lim_{f\to\infty} v_{\mathrm{a}}^{(f)}, \qquad r_{\mathrm{c}} = \lim_{f\to\infty} r_{\mathrm{c}}^{(f)}. \tag{3}$$

By locating numerically the position of the first 13 fingers we find that both quantities obey exponential laws:

$$v_{\mathrm{a}}^{(f)} = v_{\mathrm{a}} + \exp[-1.33\, f - 2.40] \qquad \text{and} \qquad r_{\mathrm{c}}^{(f)} = r_{\mathrm{c}} + \exp[-1.32\, f - 1.03], \tag{4}$$

where $v_{\rm a} \simeq 0.267949195$ and $r_{\rm c} \simeq 1.49978549$. In both cases, the magnitude of the characteristic exponent is close to 4/3. Such value was recently reported to be "universal" for a large number of rather different situations [18]. What is particularly interesting here is that our results seem to provide a novel and plausible explanation for the ubiquity of the exponent: the 4/3-scaling would arise as a consequence of the linearized dynamics near fixed points of the equations of motion.

Comparing $v_{\rm a}$ with $J = -b$ we find

$$v_{\rm a} \simeq 0.267949195\,, \tag{5}$$

$$J = 2 - \sqrt{3} \simeq 0.267949192, \tag{6}$$

a quite good agreement, corroborating the relation $\lambda_+ = 1/J$ used to derive eq. (1). The same value of $v_{\rm a}$ was obtained considering other reference lines not tangent to the basin boundary or otherwise too particularly placed. The numerical coincidence between $v_{\rm a}$ and J shown in eqs. (5) and (6) was checked for more than 30 different parameter values lying on the eigenvalue path, always yielding the same good agreement. Thus, one sees that from measurements performed exclusively in phase-space it is possible to obtain $v_{\rm a}$ and, from it, to recover the physical parameter $b = -J = -v_{\rm a}$.

We consider now the laser ring-cavity map [19]

$$z_{t+1} = \alpha\, e^{i\theta}\, z_t + \beta. \tag{7}$$

Here, $z_t = x_t + i\, y_t$ represents the complex electric field amplitude at the beginning of the t-th passage around the ring cavity, α is the reflectivity of the partially reflecting output mirror, while β is related to the laser input amplitude. The quantity θ is a complicated functional of the laser field inside the cavity and, as usual [19], we take $\theta = \Delta - \delta/(1 + |z_t|^2)$, where $\Delta = 0.4$ is the empty cavity detuning and $\delta = 6$ is the additional detuning due to the nonlinear medium. The Jacobian of the laser ring-cavity map is α^2.

In the interval $0.786 < \alpha < 0.86766$ $(0.70 < \beta < 1.22)$, we find the eigenvalue path of the laser ring-cavity map to be well approximated by

$$\beta = 5.24182 - 4.00875\alpha - 1.41208\alpha^2. \tag{8}$$

On this path, we consider the particularly interesting point $p^* = (\alpha, \beta) = (0.84753, 0.83)$. At this point there is not only a similar period-3 structure of fingers as before but also an intricate *finger-within-finger* substructure. This additional substructure forms the basin of a period-36 orbit, a surprisingly high period, multiple of the period of the basin "containing" it. Figure 3 shows the *double*-finger structure. The period-3 orbit is located at the vertices of the triangle. One of the points belonging to the period-36 orbit is $(1.022987, 0.635068)$. The unstable fixed point u is located near $(x, y) = (1.7509, -2.1924)$. At p^* we have $\alpha^2 = 0.71830$ while the dissipation measured from the period-3 fingers is 0.71828. The same dissipation rate is obtained if one considers the period-36 fingers. Similar agreement is obtained for many other parameter values obeying eq. (8). As for eq. (1), the eigenvalue path of eq. (8) may contain subintervals for which no fingers exist. For these subintervals parameter recovery becomes obviously impossible. The dissipation, however, will remain always connected to the eigenvalue. A detailed account of the structure of the eigenvalue path (surface) for the laser ring-cavity map will be presented elsewhere.

In conclusion, we have shown explicitly that physical parameters of dissipative dynamical systems with constant Jacobian may be recovered from measurements done solely on the geometrical structure of their phase-space. As is not difficult to realize, the fact that the Jacobian is a constant implies the existence of an additional constraint in the system. This constraint has the effect of introducing interdependencies between the total number of eigenvalues of the problem, *i.e.* it lowers the number of independent eigenvalues. Therefore, one sees that parameter recovery will be equally possible for any n-dimensional system having constraints of any sort that act to effectively reduce the quantity of independent eigenvalues to less than n eigenvalues.

While fingers exist over wide intervals of parameters, at present there is no theoretical recipe allowing one to anticipate their existence and location for a given set of model parameters. By investigating systematically the dynamics along eigenvalue paths one may hope to find useful clues to understand the mechanism responsible for fractal phase-spaces in physical systems. It would be very useful to find algorithms capable of predicting their ubiquitous presence.

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