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FEIGENBAUM'S CONSTANT FOR MEROMORPHIC FUNCTIONS

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We calculate Feigenbaum's constant for a double-periodic meromorphic function: the Jacobian elliptic function $sn[2K(m)x, m]$.

For $m = 0$ this function reduces to $\sin(\pi x)$, with real period, while for $m = 1$ it reduces to a hyperbolic tangent, having a pure imaginary period. For intermediary m values it is unimodal but with a non-quadratic m-dependent maximum. The bifurcation tree for $sn[2K(m)x, m]$, although very much compressed in [0, 1], presents $\delta = 4.699\ldots$ for all values of m.

Keywords: Feigenbaum's Constant; Meromorphic Functions; Meromorphic Maps.

1. Introduction

The purpose of this paper is to investigate convergence properties of a class of one-dimensional iterative maps having transcendental rather than quadratic nonlinearities. From the work of Metropolis *et al.*,[1] it is known that for a broad class of non one-to-one transformations of [0, 1] onto itself there exists a common ordering of patterns upon iteration. They observed a regular sequence of iterates (U-sequence) in maps as diverse as the logistic map

$$x_{n+1} = \lambda x_n(1 - x_n) \tag{1}$$

and the trigonometric map

$$x_{n+1} = \lambda \sin(\pi x_n)\,, \tag{2}$$

all having a common λ-ordering.[1] Subsequent work by Grossmann and Thomae[2] and Feigenbaum[3] established that under some conditions, not only the qualitative sequence of iterates is the same but certain *universal* quantitative properties as a function of the parameter[a] λ are also present. This observation is important

[a]This and some other facts about 1D maps were already known to Myrberg[4] in 1958.

because the traditional wisdom that "similar equations have similar behavior" must now be expanded to also allow for situations in which equations that are only *qualitatively* similar do indeed show a common underlying *quantitative* behavior. The quantitatively similar behavior reported by Feigenbaum using renormalization theory was the geometric convergence of the λ_i values for which Eq. (1) shows period doubling bifurcations. In other words, he showed that

$$\lim_{i \to \infty} \frac{\lambda_{i-1} - \lambda_{i-2}}{\lambda_i - \lambda_{i-1}} = \delta \tag{3}$$

where $\delta = 4.6692\ldots$ is a universal constant. Such universal scaling behavior was predicted to occur for all 1D maps of the interval [0, 1] having a quadratic maximum.[3] As shown by several workers,[5–8] the period-doubling scenario present in higher dimensional systems involves a different (also "universal") constant. For example, for period-doubling in two-dimensional maps[5–8] one finds $\delta = 8.7210\ldots$. The renormalization treatment for two-dimensional dissipative maps was done by Chen *et al.*[9] In addition, as recently shown by Alexanian,[10] it is possible to find a whole range of values for δ for some families of discontinuous two-parameter families of maps. Although not explicitly mentioned in Ref. 10, the maps discussed by Alexanian also involve meromorphic functions.

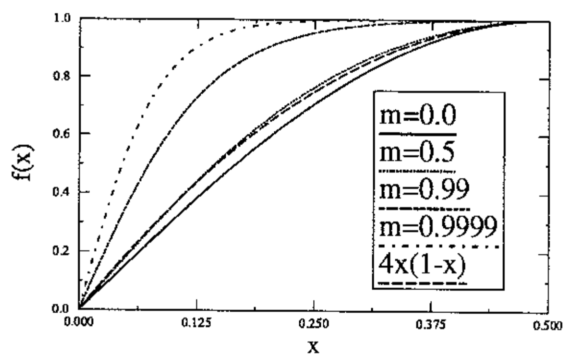

Fig. 1. The function $f(x) = sn(2Kx,\, m)$. For comparison, the logistic map is also included. The logistic map lies very close to $sn(2Kx,\, 0.5)$.

Figure 1 shows the graph of $sn[2K(m)x,\, m]$ for different values of m. $K \equiv K(m)$ is the complete elliptic integral of the first kind, corresponding to a quarter period. Note the sensible flattening of the maximum as m is changed.

The purpose of this paper is to report the calculation of δ for a class of 1D unimodal meromorphic maps having a transcendental rather than a quadratic maximum. As briefly discussed above, in principle one does not know what to expect for the numerical values of δ for such functions. The function studied here is the double-periodic Jacobian elliptic function $sn[2K(m)x,\, m]$. By changing the parameter m it is possible to substantially alter the nature of its maximum at $x = 1/2$.

As shown below, although the x-range where bifurcation phenomena takes place is greatly decreased as m is varied from the trigonometric ($m = 0$) to the hyperbolic ($m = 1$) limit, the obtained δ values do not depend on m and numerically agree with that found for maps having quadratic maximum. The *sn* function has also a negative Schwarzian derivative.

In the next section we briefly review the calculation of δ for the logistic map and present in Appendix A a FORTRAN code to calculate it, based on the superstable orbit containing the critical point $x = 1/2$. Section 3 briefly reviews some needed properties of the *sn* function, its Schwarzian derivative and gives tables showing the convergency of calculations of δ for $m = 0.0$, 0.5 and 0.99. Our conclusions are summarized in Sec. 4.

2. The logistic map

The universal scaling behavior in the period-doubling route to chaos was found by Feigenbaum while studying the logistic map defined in Eq. (1). Although the δ corresponding to Eq. (1) is theoretically a constant, its value commonly quoted in the literature shows a considerable variation on the last significant digits. Further, many sources quote values with several digits but without discussing how they were obtained. Therefore, before calculating δ for transcendental functions, we reconsider in this section the calculation of δ for the logistic map of Eq. (1). Our main motivation in doing so is to use a familiar example to assess the reliability of the algorithm employed, especially the effects of having to deal with finite-length computer words. Final computations discussed in the next sections were always done using REAL*16 precision on SUN Sparc IPC workstations and on a Cray Y-MP8/832 supercomputer with essentially the program given in Appendix A. As is easy to recognize, the precision of the method we use is solely limited by the necessity of calculating with finite word length the differences $\Delta_i = \lambda_i - \lambda_{i-1}$ between almost identical numbers.

We first attempted to calculate δ using 64 bit arithmetic on the Cray (single precision) and on SUN IPC Sparc stations. Both computers produced results of comparable quality, which converged to about 6 significant digits after 10 bifurcations. After that, the numbers quickly diverged. We advise the reader to run the program given in the Appendix on different computers and to compare the numbers obtained with the "reference" ones given in Table 1. The result of all this is that longer computer words are essential to assure convergence or, at least, to postpone divergence to a later stage. We believe that the double precision calculations reported here are guaranteed to 10 converged digits. It would be interesting to extrapolate the numbers given in the table using, e.g. Aitken's procedure. However we will not attempt this here since the numbers obtained are by far more precise than any possible experimental determination of them. Table 1 shows the convergence of the calculation of δ for the quadratic map generated in a time equivalent to 18390 seconds on a Cray Y-MP8/832 supercomputer. The several δ_i were obtained

Table 1. Evolution of δ_i for the logistic map using 128-bits precision.

i	λ_i	$\Delta_i = \lambda_i - \lambda_{i-1}$	$\delta_i = \frac{\Delta_{i-1}}{\Delta_i}$
1	3.23606797749978969		
2	3.49856169932770151	0.26249372182791182	
3	3.55464086276882486	0.05607916344112335	4.68077099801069538
4	3.56666737985626851	0.01202651708744365	4.66295961111410258
5	3.56924353163711033	0.00257615178084182	4.66840392591840023
6	3.56979529374994462	0.00055176211283428	4.66895374096762278
7	3.56991346542234851	0.00011817167240389	4.66915718132884348
8	3.56993877423330548	0.00002530881095697	4.66919100248509615
9	3.56994419460806493	0.00000542037475945	4.66919947054772577
10	3.56994535548646858	0.00000116087840365	4.66920113460104223
11	3.56994560411107843	0.00000024862460986	4.66920150951355232
12	3.56994565735885649	0.00000005324777806	4.66920158752238550
13	3.56994566876289996	0.00000001140404347	4.66920160451218518
14	3.56994567120529685	0.00000000244239689	4.66920160811593520
15	3.56994567172838347	0.00000000052308662	4.66920160889206914
16	3.56994567184041260	0.00000000011202914	4.66920160905775824
17	3.56994567186440581	0.00000000002399321	4.66920160909331230
18	3.56994567186954443	0.00000000000513861	4.66920160910092442
19	3.56994567187064496	0.00000000000110053	4.66920160910218325
20	3.56994567187088066	0.00000000000023570	4.66920160910513033
21	3.56994567187093114	0.00000000000005048	4.66920160908967193
22	3.56994567187094195	0.00000000000001081	4.66920160915374310
23	3.56994567187094427	0.00000000000000232	4.66920160858269040
24	3.56994567187094476	0.00000000000000050	4.66920161186166765
25	3.56994567187094487	0.00000000000000011	4.66920159077937689

Table 2. Evolution of δ_i for $m = 0$, i.e., for the trigonometric map of Eq. (2).

i	λ_i	$\Delta_i = \lambda_i - \lambda_{i-1}$	$\delta_i = \frac{\Delta_{i-1}}{\Delta_i}$
1	0.77773376617160609		
2	0.84638217170667942	0.06864840553507333	
3	0.86145035088263241	0.01506817917595299	4.55585274992136916
4	0.86469418074586915	0.00324382986323674	4.64518171767422836
5	0.86538967340501526	0.00069549265914612	4.66407491233524524
6	0.86553866160451903	0.00014898819950377	4.66810567187598369
7	0.86557057191975869	0.00003191031523966	4.66896670825060651
8	0.86557740620568342	0.00000683428592472	4.66915133360485230
9	0.86557886990381922	0.00000146369813581	4.66919083759928510
10	0.86557918338329202	0.00000031347947280	4.66919930270159509
11	0.86557925052100254	0.00000006713771051	4.66920111507739019
12	0.86557926489984450	0.00000001437884196	4.66920150330595022
13	0.86557926797935214	0.00000000307950764	4.66920158644349981
14	0.86557926863888838	0.00000000065953623	4.66920160425014605
15	0.86557926878014084	0.00000000014125246	4.66920160806364458
16	0.86557926881039279	0.00000000003025195	4.66920160888039613
17	0.86557926881687184	0.00000000000647904	4.66920160905531751
18	0.86557926881825945	0.00000000000138761	4.66920160909278087
19	0.86557926881855663	0.00000000000029718	4.66920160910079280
20	0.86557926881862028	0.00000000000006365	4.66920160910236042

by using the full 128 bit representation of the Δ_i. The truncated Δ_i displayed in Table 1 are only intended to stress the speed with which their number of significant digits is reduced as the λ_i become closer and closer. The steady convergency of the δ_i is interrupted at $i = 20$, where they start to oscillate incoherently. We regard this spurious effect as loss of precision due to our using of a computer word having only finite length. Table 2 presents similar results obtained using the trigonometric map evaluated with the intrinsic trigonometric sine function of the Cray. The generation of Table 2 required 12664 seconds. For comparison we quote that the first 23 lines of Table 1 were obtained in 4961 seconds. The difference of 13429 seconds corresponds to the time needed to find the last two values ($i = 24$ and 25). It is important to notice that due to the particular nature of the recurrence relation involved in the iteration of the map, it is unfortunately not possible to vectorize the program.

3. Jacobian Elliptic Functions

As demonstrated by Singer[11] in 1978, an important tool to study 1D dynamical systems is the Schwarzian derivative. It can be used to establish an upper bound on the number of attracting periodic orbits that the map characterizing the 1D dynamics might have. The Schwarzian derivative of a function f at x is defined as

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 .$$

A unimodal map of the interval with negative Schwarzian derivative having n critical points may have at most $n + 2$ stable periodic orbits at a given parameter value. For $f(x) = \lambda x(1-x)$ one finds $Sf = -6/(1-2x)^2$. Therefore $Sf < 0$ for all x, including the critical point $x_c = 1/2$ at which $Sf \to -\infty$. For $f(x) = \lambda \sin(\pi x)$, $Sf = -\pi^2 - \frac{3}{2}\pi^2\, tg^2(\pi x)$, which is also negative for all x. For the function $f(x) = \lambda\, sn[2K(m)x, m]$ one finds

$$\begin{aligned} f' &= 2K\lambda\, cn(2Kx,\, m)\, dn(2Kx,\, m)\,, \\ f'' &= (2K)^2\lambda[2m\, sn^3(2Kx,\, m) - (1+m)\, sn(2Kx,\, m)]\,, \\ f''' &= (2K)^3\lambda\, cn(2Kx,\, m)\, dn(2Kx,\, m)[6m\, sn^2(2Kx,\, m) - (1+m)]\,, \end{aligned}$$

and, accordingly,

$$Sf = -2K^2 \frac{2m(1+m)\, sn^4(2Kx,\, m) + (m^2 - 10m + 1)\, sn^2(2Kx,\, m) + 2 + 2m}{[m\, sn(2Kx,\, m) - 1][sn(2Kx,\, m) + 1][sn(2Kx,\, m) - 1]}\,,$$

which can be shown to be negative for all x. For $m = 0$, $sn(2Kx,\, m) \equiv \sin(\pi x)$ and the above equations correctly reproduce the aforementioned trigonometric limit. In the unit interval the first derivative is zero at the critical point $x_c = 1/2$, the root of $cn(2Kx,\, m) = 0$. $dn(2Kx,\, m) = 0$ has no solution for real x. The location of the zero of the first derivative does not depend on m. Therefore one sees that sn behaves much like the $m = 0$ trigonometric case.

For $m \neq 0$ we computed the elliptic integral K using the R_F algorithm of Carlson[12] with a tolerance of 10^{-20}. Test runs for other values showed the numbers to be independent of this value. The *sn* function was computed using the arithmetic-geometric mean procedure described in Sec. 16.4 of Abramowitz and Stegun.[13]

Table 3 presents the evolution of the bifurcation parameters for $m = 0.5$. As seen from Fig. 1, this case roughly approximates the quadratic map although both functions do indeed cross over each other. A big difference in the bifurcation tree is the range of parameters: the doubling range of the elliptic function is very much reduced when compared to that of the quadratic map. Table 3 needed 19894 seconds to be generated.

Table 4 shows as before the sequence of bifurcations for $m = 0.99$. This table was generated in 19843 seconds. Although this time is comparable to that needed

Tables 3 and 4. δ_i for $m = 0.5$ (top) and $m = 0.99$ (bottom).

i	λ_i	$\Delta_i = \lambda_i - \lambda_{i-1}$	$\delta_i = \frac{\Delta_{i-1}}{\Delta_i}$
1	0.81551353094105212		
2	0.87830989596184075	0.06279636502078863	
3	0.89154012214861947	0.01323022618677871	4.74643170375594721
4	0.89436898661401333	0.00282886446539386	4.67686817400658521
5	0.89497467795036901	0.00060569133635568	4.67047206323694686
6	0.89510439100116956	0.00012971305080055	4.66947105643997452
7	0.89513217123496903	0.00002778023379948	4.66925698814642747
8	0.89513812089496696	0.00000594965999793	4.66921367088871580
9	0.89513939512924592	0.00000127423427896	4.66920416141351210
10	0.89513966803117552	0.00000027290192959	4.66920215940360301
11	0.89513972647841031	0.00000005844723479	4.66920172649196159
12	0.89513973899601810	0.00000001251760780	4.66920163430181544
13	0.89513974167690637	0.00000000268088826	4.66920161449263735
14	0.89513974225107054	0.00000000057416417	4.66920161025817430
15	0.89513974237403892	0.00000000012296838	4.66920160935028575
16	0.89513974240037498	0.00000000002633606	4.66920160915596793
17	0.89513974240601536	0.00000000000564038	4.66920160911433665

i	λ_i	$\Delta_i = \lambda_i - \lambda_{i-1}$	$\delta_i = \frac{\Delta i-1}{\Delta_i}$
1	0.91734829278495591		
2	0.96316302308513360	0.04581473030017769	
3	0.96851781958737315	0.00535479650223955	8.55583032539452482
4	0.96944798505559518	0.00093016546822203	5.75682143143309075
5	0.96964068512591296	0.00019270007031778	4.82701156615094476
6	0.96968171469545478	0.00004102956954182	4.69661447755065371
7	0.96969049113882027	0.00000877644336549	4.67496545390423381
8	0.96969237029607514	0.00000187915725487	4.67041453967500116
9	0.96969277273153529	0.00000040243546015	4.66946241307690417
10	0.96969285891985969	0.00000008618832440	4.66925726849623642
11	0.96969287737871223	0.00000001845885254	4.66921355050030824
12	0.96969288133203102	0.00000000395331879	4.66920416386379859
13	0.96969288217871074	0.00000000084667972	4.66920215658341549
14	0.96969288236004360	0.00000000018133286	4.66920172631557852
15	0.96969288239887955	0.00000000003883595	4.66920163421138538
16	0.96969288240719702	0.00000000000831747	4.66920161447981251

to generate Table 3, the present table contains only 16 values. Further, note that the interval where the doublings occur is extremely compressed to the region near the right end of the [0, 1] domain.

4. Conclusions

We report an accurate numerical evaluation of Feigenbaum's constant for a class of meromorphic double-periodic functions, namely for the Jacobian elliptic function $sn(2Kx, m)$. The main motivation for looking at this function is that although being unimodal in [0, 1], it presents transcendental rather than quadratic maxima as a function of m. The map continuously changes from a trigonometric (for $m = 0$) to a hyperbolic (for $m = 1$) limit. We found that in contrast with its trigonometric limit, the Schwarzian derivative of sn has more than one singularity in [0, 1]. sn presents a relatively wide transcendental maximum at higher m values. However, in contrast with other two-parameter meromorphic functions recently studied,[10] sn presents the same Feigenbaum constant as the logistic map. The numerical results presented here should be useful in investigations of multiple scaling and of the fine structure of period-doubling.[14] It would be of interest to investigate Feigenbaum's constant for other Jacobian elliptic functions, especially those suitably chosen to present discontinuities on, say, the [0, 1] interval.

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References

1. N. Metropolis, M. L. Stein and P. R. Stein, *J. Comb. Theory* **15**, 25 (1973).
2. S. Grossmann and S. Thomae, *Z. Naturf.* **32a**, 1353 (1977).
3. M. Feigenbaum, *J. Stat. Phys.* **19**, 25 (1978); **21**, 669 (1979).
4. See, for example, R. May, *Nature* **261**, 459 (1976); further aspects are discussed in Chapter 3 of the book by C. Mira, *Chaotic Dynamics, from 1D Endomorphism to 2D Diffeomorphism* (World Scientific, 1987).
5. G. Bennetin, C. Cercignani, L. Galgani, and A. Giorgilli, *Lett. Nuovo Cim.* **28**, 1 (1980).
6. P. Collet, J. P. Eckmann and H. Koch, *Physica* **3D**, 457 (1981).
7. J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, *Physica* **3D**, 468 (1981).
8. T. C. Bountis, *Physica* **3D**, 577 (1981).
9. C. Chen, G. Györgi and G. Schmidt, *Phys. Rev.* **A34**, 2568 (1986); **36**, 5502 (1987).
10. M. Alexanian, *Physica* **181A**, 53 (1992).
11. D. Singer, *SIAM J. Appl. Math.* **35**, 260 (1978).
12. B.C. Carlson, *Num. Math.* **33**, 1 (1979).

13. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, NY, 1972), p. 571.
14. J. Mao and B. Hu, *Int. J. Mod. Phys.* **B2**, 65 (1988).

Appendix A

Program used to calculate the numbers given in Table 1.

```
      program delta
      implicit real*16(a-h,k-z)
      parameter( xzero=0.5d0,  jstop=25 )
      dimension savel(50),saved(50),dif(50)
      data lambda/3.0d0/, step0/1.0d0/
      open(17,file='delta.out')
      do 999 j=1,jstop
      sinal = 1.0d0
      if( mod(j,2).eq.0 ) sinal = -1.0d0
      step0 = step0/4.669d0
      step  = step0
      iter  = 2
      do 10 i=1,j-1
 10      iter = iter*2
      lambda= lambda + step
777   continue
      x     = xzero
      do 20 i=1,iter
 20      x = lambda*x*(1.0d0-x)
      if( (x-xzero)*sinal .lt. 0.0d0 ) then
            lambda = lambda - step
            step   = step/2.0d0
            endif
      lambda= lambda + step
      if( lambda-step .lt. lambda) go to 777
      savel(j) = lambda
      saved(j) = 0.0d0
      dif  (j) = 0.0
      if( j.gt.1 ) dif(j)   = savel(j)-savel(j-1)
      if( j.gt.2 ) saved(j) = dif(j-1)/dif(j)
      write(17,30) j,savel(j),dif(j),saved(j)
 30   format(1x,i3,2x,3(2x,f20.17))
999   continue
      stop
      end
```