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Constants of motion for the KdV and mKdV equations *

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We present recurrence relations that generate the terms appearing in the known expressions for the densities and the fluxes of the Korteweg–de Vries equation, and in the densities of the modified Korteweg–de Vries equation. Our relations are based on the observation that for a given rank r the set $\{P_{2r}\}$ of all partitions of the integer $2r$ contains all monomials of $\{X_{r-1}\}$ and $\{T_r\}$. The relations are very easy to program in systems able to perform computer algebra. In addition, we report three new constants of motion for the modified Korteweg–de Vries equation.

It is well known that certain nonlinear partial differential equations arising in the study of a number of different physical systems ranging from nonlinear optics to hadron physics obey what are called *conservation laws*. A prominent example is the Korteweg–de Vries (KdV) equation which contains an infinite sequence of conservation laws [1]. The discovery of such an infinite sequence of conservation laws for the KdV equation motivated, in subsequent years, a lot of activity on evolution equations possessing infinitely many symmetries [2]. Conservation laws are equations of the generic type

$$T_t + X_x = 0, \tag{1}$$

where T (the conserved *density*) and $-X$ (the corresponding *flux* of T) are polynomials in a "field" variable $u(x, t)$ and its derivatives, i.e. are sums of *monomials*

$$c(i_0, i_1,\dots, i_n)u_0^{i_0}u_1^{i_1}\dots u_n^{i_n}, \tag{2}$$

where $u_0 \equiv u(x, t)$, $u_n \equiv \partial^n u(x, t)/\partial x^n$, $c(i_0, i_1,\dots, i_n)$ is a constant and i_0, i_1, ... are nonnegative integers. A *brute-force* approach to obtain densities and fluxes consists of summing several monomials (2) and, using eq. (1), setting up a system of equations to be solved for the several (sometimes hundreds) of $c(i_0, i_1,\dots, i_n)$. For systems with *uniform rank* [5] there are ways of knowing how many monomials are needed in the summation for every rank r. In practice one starts with far more monomials than needed and from constraints imposed by eq. (1) determines those monomials really present in T and X. This approach is, however, difficult to follow since one quickly ends up with systems of equations involving several hundreds of coefficients $c(i_0, i_1,\dots, i_n)$ to be determined. In this paper we present recurrence relations that provide directly the monomials that really appear in T and X and only these. This has the effect of reducing the number of unknowns and equations to a minimum, thereby rendering possible the investigation of conservation laws of the KdV and mKdV equations and, we hope, in the future, of other nonlinear evolution equations. It was pointed out to us by the referee of this paper that recurrence relations have long been used in connection with conservation laws for soliton equations [3]. We see, however, no direct connection between our results and the already available body of results on recurrence relations for soliton equations.

In a recent paper, Torriani [4] showed how to use combinatorial methods to obtain constants of motion for the Korteweg–de Vries (KdV) and related equations. His very interesting procedure

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consists of associating partitions of integers and their Ferrers graphs to the first density T and the first flux $-X$ and then, through simple rules, to generate all subsequent T and X obeying eq. (1). For example, let us take $\cdot\cdot$ as the Ferrers graph associated with u_0. His first conjecture [4] gives then :: as the only possible graph. This graph is associated to the partition 2^2 and to the monomial u_0^2, which is the only one present in T_2. Applying Conjecture 1 to the graph :: gives

$$\vdots \qquad \text{and} \qquad ::: \;,$$

which are associated to the partitions 2^3 and 3^2, corresponding to u_0^3 and u_1^2, respectively. A further application of the conjecture generates three different graphs, corresponding to the partitions 2^4, 23^2 and 4^2, producing u_0^4, $u_0 u_1^2$ and u_2^2, respectively. In this clever way Torriani was able to generate all known densities and fluxes of the KdV equation as well as the known densities of the modified KdV equation (mKdV).

The main purpose of this paper is to present an alternative method to generate the monomials present in the densities and fluxes of the KdV and mKdV equations. Our approach has much to do with the combinatorial approach proposed by Torriani [4], but the overlap of both procedures is difficult to assess. The biggest advantage of the method being proposed here is that it provides *recurrence relations* from which the monomials can be obtained. These recurrence relations are easy to program in systems able to perform algebraic computations like, for example, REDUCE.

Let us start with the KdV equation. We use the word *monomial* as in Torriani [4], but denote the set of monomials in T_r by $\{T_r\}$, in X_r by $\{X_r\}$, etc. Our method is based on the following *empirical* observations made of the set of monomials presently available in the literature [1,5]:

(a) for a given rank r, the set $\{P_{2r}\}$ of all partitions of the integer $2r$ contains all monomials of $\{X_{r-1}\}$ and $\{T_r\}$;

(b) those monomials belonging to $\{P_{2r}\}$ but not to $\{X_{r-1}\}$ define a set $\{Q_{2r}\}$; those in $\{P_{2r}\}$ but not in $\{T_r\}$ define a set $\{R_{2r}\}$;

(c) the set $\{P_i\}$ may be easily generated recursively from integrations over u and in the derivatives of u;

(d) all three sets are obtained from identical recurrence relations, but with different initial conditions, i.e. the generation of $\{Q_i\}$ and $\{R_i\}$ is identical to that of $\{P_i\}$.

Since the method generates only monomials (and not the numerical coefficients in the densities and fluxes), all integrations over u_j may be performed as though they were simple multiplications by u_j. Explicitly, for $r \geq 2$ we obtain

$$\{T_r\} = \{P_{2r}\} - \{R_{2r}\}, \tag{2a}$$

$$\{X_r\} = \{P_{2r+2}\} - \{Q_{2r+2}\}, \tag{2b}$$

where

$$P_i = u_{i-2} + \sum_{k=0}^{[(i-4)/2]} u_k P_{i-k-2}, \quad i \geq 4, \tag{3}$$

$$Q_i = u_{i-2} + \sum_{k=0}^{[(i-7)/2]} u_k Q_{i-k-2}, \quad i \geq 7, \tag{4}$$

$$R_i = u_{i-2} + \sum_{k=0}^{[(i-5)/2]} u_k R_{i-k-2}, \quad i \geq 5, \tag{5}$$

where the symbol $[x]$ means the largest integer not greater than x. The following initial values are required by the recurrence relations: $P_2 = Q_2 = R_2 = u_0$, $P_3 = Q_3 = R_3 = u_1$, $Q_4 = R_4 = u_2$, $Q_5 = u_3$ and $Q_6 = u_4$. Although numerical coefficients of the monomials in eqs. (3–4) are totally meaningless, we found it convenient to avoid summation of repeated terms. This can be achieved by dropping from the summations all products of u_k with monomials containing u_j with $j < k$. This obviously constrains all numerical coefficients to be unity. In appendix A we give a REDUCE program that was used here to generate the monomials in the densities and fluxes of the invariants of the KdV equation. Procedure PROD in this program is used to avoid summation of repeated terms. The monomials will also be correctly generated if calls to PROD(J, F(K)) are simply replaced by U(J)*F(K). However, in this case the numerical coefficients of the monomials will not be unity anymore.

The above recurrence relations for T_r and X_r define our method to generate all and only those monomials contained in the densities and fluxes, respectively, for the KdV equation. In an analogous way, the densities for the modified KdV equation can be obtained from

$$T_r = v_{r-1}^2 + \sum_{k=0}^{r-2} v_k^2 T_{r-k-1} + R_r, \quad r \geq 2, \tag{6}$$

where $T_1 = v_0^2$, $R_r = 0$ for $r \leq 4$ and

$$R_r = v_{r-3}^2 v_0 v_2 + (1-\delta_{r,5})(1-\delta_{r,6}) \times v_{r-4}^2 \big(v_1 v_3 + (1-\delta_{r,7}) v_0 v_4 \big), \quad r \geq 5,$$

δ_{ij} being the Kroenecker delta function. Appendix B gives the corresponding REDUCE implementation of the above relation.

For the KdV equation Miura, Gardner and Kruskal [1] reported T_r for $r \leq 10$ and X_r for $r \leq 7$. For the mKdV, besides $T_{1/2}$ and $X_{1/2}$, they reported T_r for $r \leq 5$, together with X_1 and X_2. In a subsequent paper [5] they reported T_{11} for the KdV equation. It may be checked that our recurrence relations, and the programs given in appendices A and B, do reproduce all and only the known monomials.

To give an idea of the amount of work involved in the determination of each invariant, table 1 presents the number of unknowns (i.e. monomials) in T_r and X_r for $r \leq 10$. This table compares the number of monomials that must be considered in a brute-force calculation as mentioned before with the *effective* number of monomials as predicted by our recurrence relations. Obviously, a lesser number of monomials implies a smaller system of equations to be solved. Table 1 also shows the number of equations that one needs to solve when adopting our relations instead of brute force. Following ref. [1], we express all coefficients in T_r in terms of $c(r, 0, \ldots, 0)$ and arbitrarily set $c(r, 0, \ldots, 0) = 1/r$. Accordingly, the total number of unknowns is given by the sum of monomials in T_r and in X_r minus 1.

As part of the present work we also recalculated for the KdV equation all constants of motion up to $r \leq 10$, and for the mKdV up to $r \leq 8$. The constants of motion were obtained from the algebraic procedures given in the appendices A and B. After determining the monomials for every rank, we calculated all numerical coefficients by solving the linear system of equations resulting from eq. (1). Such a calculation guarantees that every function generated from the monomials obtained by our method was indeed a constant of motion and that our method generates the correct number of monomials. We found a misprint in one of the constants of motion already available in the literature: in eq. (10b) of ref. [5] the coefficient of $u_0 u_4^2$ should be $+144/7$ instead of $-144/7$.

For the mKdV we obtained the following three new constants of motion:

$$\begin{aligned} T_6 = {} & \tfrac{1}{12} v_0^{12} - \tfrac{55}{2} v_0^8 v_1^2 + 66 v_0^6 v_2^2 - 319 v_0^4 v_1^4 - \tfrac{594}{7} v_0^4 v_3^2 \\ & + \tfrac{2640}{7} v_0^3 v_2^3 + \tfrac{12276}{7} v_0^2 v_1^2 v_2^2 + \tfrac{396}{7} v_0^2 v_4^2 - \tfrac{3960}{7} v_0 v_2 v_3^2 \\ & - \tfrac{5214}{35} v_1^6 - \tfrac{3564}{7} v_1^2 v_3^2 + \tfrac{2178}{7} v_2^4 - \tfrac{108}{7} v_5^2, \end{aligned} \tag{7}$$

$$\begin{aligned} T_7 = {} & \tfrac{1}{14} v_0^{14} - 39 v_0^{10} v_1^2 + 117 v_0^8 v_2^2 - 1014 v_0^6 v_1^4 \\ & - \tfrac{14004}{7} v_0^6 v_3^2 + \tfrac{9360}{7} v_0^5 v_2^3 + \tfrac{63180}{7} v_0^4 v_1^2 v_2^2 + \tfrac{1404}{7} v_0^4 v_4^2 \\ & - \tfrac{28080}{7} v_0^3 v_2 v_3^2 - \tfrac{25740}{7} v_0^2 v_1^6 - \tfrac{42120}{7} v_0^2 v_1^2 v_3^2 \\ & + \tfrac{35100}{7} v_0^2 v_2^4 - \tfrac{8424}{77} v_0^2 v_5^2 + \tfrac{179712}{7} v_0 v_1^2 v_2^3 \\ & + \tfrac{16848}{11} v_0 v_2 v_4^2 + \tfrac{96876}{7} v_1^4 v_2^2 + \tfrac{109512}{77} v_1^2 v_4^2 \\ & - \tfrac{39312}{11} v_1 v_3^3 - \tfrac{598104}{77} v_2^2 v_3^2 + \tfrac{1944}{77} v_6^2, \end{aligned} \tag{8}$$

Table 1

Number of monomials (unknowns) in T_r and X_r with rank $r \leq 10$. BF represents the number of all monomials that exist for a given rank while M gives the effective number of monomials present in T_r or X_r as generated by our method. The last two lines give the total number of unknowns and of equations in the system obtained by substituting T_M and X_M into eq. (1). For $r > 5$ the system is overdetermined.

Rank r	1	2	3	4	5	6	7	8	9	10
Coef. in T_{BF}	1	2	4	7	12	21	34	55	88	137
Coef. in T_M	1	1	2	3	4	7	10	14	22	32
Coef. in X_{BF}	2	4	7	12	21	34	55	88	137	210
Coef. in X_M	2	3	5	8	13	20	31	47	71	105
No. of unknowns	2	3	6	10	16	26	40	60	92	136
No. of equations	2	3	6	10	16	27	42	64	99	148

$$
\begin{aligned}
T_8 = {} & \tfrac{1}{16} v_0^{16} - \tfrac{105}{2} v_0^{12} v_1^2 + 189 v_0^{10} v_2^2 - \tfrac{5145}{2} v_0^8 v_1^4 \\
& - 405 v_0^8 v_3^2 + 3600 v_0^7 v_2^3 + 31860 v_0^6 v_1^2 v_2^2 \\
& + 540 v_0^6 v_4^2 - 16200 v_0^5 v_2 v_3^2 - 30474 v_0^4 v_1^6 \\
& - 34020 v_0^4 v_1^2 v_3^2 + 31590 v_0^4 v_2^4 - \tfrac{4860}{11} v_0^4 v_5^2 \\
& + 286848 v_0^3 v_1^2 v_2^3 + \tfrac{136080}{11} v_0^3 v_2 v_4^2 + 330804 v_0^2 v_1^4 v_2^2 \\
& + \tfrac{197640}{11} v_0^2 v_1^2 v_4^2 - \tfrac{571536}{11} v_0^2 v_1 v_3^3 - \tfrac{1507896}{11} v_0^2 v_2^2 v_3^2 \\
& + \tfrac{29160}{143} v_0^2 v_6^2 - \tfrac{3617136}{11} v_0 v_1^2 v_2 v_3^2 + \tfrac{719280}{11} v_0 v_2^5 \\
& - \tfrac{544320}{143} v_0 v_2 v_5^2 + \tfrac{489888}{143} v_0 v_4^3 - \tfrac{63153}{7} v_1^8 \\
& - \tfrac{712476}{11} v_1^4 v_3^2 + \tfrac{3083508}{11} v_1^2 v_2^4 - \tfrac{515160}{143} v_1^2 v_5^2 \\
& + \tfrac{5334336}{143} v_1 v_3 v_4^2 + \tfrac{3893832}{143} v_2^2 v_4^2 - \tfrac{2267028}{143} v_3^4 \\
& - \tfrac{5832}{143} v_7^2 .
\end{aligned}
\tag{9}
$$

The corresponding fluxes contain many more terms than the densities. For example, X_6 contains 42 terms, X_7 contains 69 and X_8 contains 110, while T_6, T_7 and T_8 above contain 13, 20 and 32 terms, respectively.

In summary, we report simple recurrence relations able to generate all monomials in the known expressions for the densities and fluxes of the KdV equation, as well as those in the densities of the mKdV equation. The great advantage of our recurrence relations is that they are easy to implement in systems able to perform computer algebra. A REDUCE implementation of them is given in appendices A and B. Therefore, besides allowing direct construction of constants of motion, one is able to investigate peculiar properties of complicated nonlinear evolution equations as well as to demonstrate the strong impact of combinatorial analysis in this field. The new constants of motion reported in this paper support the conjectures of Torriani [4]. In a subsequent paper we intend to present a recurrence relation for the fluxes of the mKdV equation and to investigate the structure of the next few constants of motion for both KdV and mKdV equations. All this work is part of a preliminary effort the ultimate goal of which is the study of the integrability of much more complicated nonlinear evolution equations of the generic type

$$u_t - a u_{xxt} = F(u,\ u_x,\ u_{xx},\ \ldots), \tag{10}$$

like, for example, the BBM equation [6] and the equations discussed by Caldas and Tasso [7]. Before concluding we would like to observe that for the much simpler case discussed in this paper (of equations having uniform rank and $a = 0$) there is already in the literature [8] an algorithm in RLISP to generate the densities (not the fluxes). It would be interesting now to write a REDUCE code able of generating not only the monomials but also the corresponding numerical coefficients.

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Appendix A

```
LINELENGTH 60$ OFF NAT$ ON LIST$ OPERATOR U$
ARRAY P(50), Q(50), R(50), TT(20), XX(20)$

COMMENT     PROGRAM TO GENERATE DENSITIES TT AND FLUXES XX
            FOR THE KDV EQUATION UP TO RANK R (= RNK)
            USING EQS. (2-5) OF THE TEXT$

RNK: = 15$ KLIM := 2*(RNK + 1)$
Q(2) := U(0)$ Q(3) := U(1)$ Q(4) := U(2)$ Q(5) := U(3)$ Q(6) := U(4)$
P(2) := U(0)$ P(3) := U(1)$
R(2) := U(0)$ R(3) := U(1)$ R(4) := U(2)$
```

```
PROCEDURE PROD(J,POLIN)$
BEGIN SCALAR TERMO,PROV$ ARRAY G(25), V(25, 25), VV(25)$
        TERMO := POLIN$
        IF J EQ 0 THEN PROV := U(0)*TERMO$
        IF J NEQ 0 THEN BEGIN$
             G(0) := COEFF(TERMO, U(0), VV)$
             FOR I := 0 : G(0) DO V(0, I) := VV(I)$
             IF J GEQ 2 THEN BEGIN$
                FOR I := 1 : J - 1 DO BEGIN$
                      G(I) := COEFF(V(I - 1, 0), U(I), VV)$
                      FOR K := 0 : G(I) DO V(I, K) := VV(K)$
                      END$
                END$
             PROV := U(J)*V(J - 1, 0)$
             END$ RETURN PROV$
END$

COMMENT      XX(K) = P(2*(K + 1)) - Q(2*(K + 1))
             TT(K) = P(2*K) - R(2*K)$

FACTOR U(0), U(1), U(2), U(3), U(4), U(5), U(6), U(7), U(8), U(9), U(10),
          U(11), U(12), U(13), U(14), U(15)$

FOR K := 4: KLIM DO BEGIN$
      IF K GEQ 4 THEN
           P(K) := U(K - 2) + FOR J := 0: ((K - 4)/2) SUM PROD (J, P(K - J - 2))$
      IF   K GEQ 5 THEN
           R(K) := U(K - 2) + FOR J := 0: ((K - 5)/2) SUM PROD (J, R(K - J - 2))$
      IF   K GEQ 7 THEN
           Q(K) := U(K - 2) + FOR J := 0: ((K - 7)/2) SUM PROD (J, Q(K - J - 2))$
      IF   FIXP(K/2 - 1) THEN BEGIN$
           XX(K/2 - 1) := P(K) - Q(K)$
           WRITE "XX(",K/2 - 1,") := ", XX(K/2 - 1)$
           END$
      IF   FIXP(K/2) THEN BEGIN$
           TT(K/2) := P(K) - R(K)$
           WRITE "TT(",K/2,")  := ", TT(K/2) END$
      END$
END$
```

APPENDIX B

```
LINELENGTH 60$ OFF NAT$ ON LIST$ OPERATOR V$ ARRAY MT(50), MR(50)$

COMMENT      PROGRAM TO GENERATE THE DENSITIES MT FOR THE
             MODIFIED KDV EQUATION UP TO RANK R (= RNK)
             USING EQ. (6) OF THE TEXT$
```

```
RNK := 8$ MT(1) := V(0)**2$
MR(1) := 0$ MR(2) := 0$ MR(3) := 0$ MR(4) := 0$ MR(5) := V(0)*V(2)**3$
MR(6) := V(0)*V(2)*V(3)**2$ MR(7) := V(0)*V(2)*V(4)**2 + V(1)*V(3)**3$
FACTOR V(0), V(1), V(2), V(3), V(4), V(5), V(6), V(7), V(8), V(9), V(10),
        V(11), V(12), V(13), V(14), V(15)$
FOR K := 2: RNK DO BEGIN$
    IF K GEQ 8 THEN
      MR(K) := V(K - 3)**2*V(0)*V(2) + V(K - 4)**2*(V(1)*V(3) + V(0)*V(4))$
    IF K GEQ 2 THEN
      MT(K) := V(K - 1)**2 + FOR J := 0: (K - 2) SUM (V(J)**2*MT(K - J - 1)) + MR(K)$
    WRITE "MT(", K,") := ", MT(K)$
    END$
END$
```

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