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Equivalence among orbital equations of polynomial maps

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This paper shows that orbital equations generated by iteration of polynomial maps do not necessarily have a unique representation. Remarkably, they may be represented in an infinity of ways, all interconnected by certain nonlinear transformations. Five direct and five inverse transformations are established explicitly between a pair of orbits defined by cyclic quintic polynomials with real roots and minimum discriminant. In addition, infinite sequences of transformations generated recursively are introduced and shown to produce unlimited supplies of equivalent orbital equations. Such transformations are generic and valid for arbitrary dynamics governed by algebraic equations of motion.

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1. Introduction

Periodic dynamics of classical oscillators are commonly studied numerically when the period grows. By contrast, the exact investigation of periodic orbits, feasible for systems governed by algebraic equations of motion, is rarely reported because of the inherent theoretical complications. This paper describes an exploration of exact periodic equations of motion which arise in the investigation of dynamics generated by polynomial maps.

Many applications, especially in biology and theoretical physics, can be usefully described by studying the algebraic properties of the equations of motion obtained by composition of polynomial maps. Whereas numerical work provides immediate access to dynamical processes, there is also great merit in exact algebraic work in that it opens the possibility of uncovering the systematics behind recurring regularities and obtaining valuable information buried behind the regular self-similar repetition of structures typically present in phase diagrams. Of particular interest is the prediction of metric properties of the accumulations of doubling and adding cascades observed abundantly in applications. For instance, there are a number of recent surveys concerning the complexities observed in accumulations of doubling and adding cascades in laser systems, see Ref. 1, in chemistry,[2] in biochemical models,[3,4] and in the dynamics of cancer.[5]

The rationale for considering equations of motion generated by repeated iteration is the fact that equations of higher degree are expected to comprehend equations of lower degree, so that the solution of higher degree equations should involve a methodology similar to the one used for equations of lower degrees. Recall that equations of motion generated by iteration are necessarily Abelian equations, meaning that they can be solved algebrically.[6,7] Basically, the goal is to explore group properties of Abelian equations generated by iteration to understand and predict the sprouting of stability in phase diagrams, to articulate a theoretical framework that could accommodate the insight won with numerical computations.

This paper reports the discovery of infinite sequences of transformations interconnecting orbital equations of motion of arbitrary degrees generated by compositions of (i) the quadratic map, (ii) the Hénon map and (iii) the canonical quartic map, all defined in Sec. 3 below. Specifically, we show that a certain transformation found to establish the isomorphism between two totally real cyclic quintic fields of minimum discriminant is surprisingly not unique and that four additional transformations exist that establish the same isomorphism. Moreover, we determine explicitly the five inverse transformations, an important question not addressed in previous works. The combined actions on orbital points of the direct, inverse, and composed transformations are characterized. In addition, generic families of transformations generated recursively are introduced and used to produce an unlimited supply of minimum discriminant isomorphic orbits. Such transformations are not restricted to the illustrative examples discussed here, but are valid generically for any dynamical systems of algebraic origin.[8–10] They are important for the study of orbit proliferation in equivalence classes of equations of motion produced by discrete maps.

2. The Known Equivalence and Minimum Discriminants

To motivate the introduction of the generic transformation chains discussed in Sec. 4, we start by considering a concrete example. In 1955, in a pioneering application of computers to algebraic number theory, Cohn[11] produced three tables of irreducible quintic polynomials with integral coefficients, arranged by increasing order of their discriminant, for the three signatures[12] (n, ℓ), namely $(1, 2)$, $(3, 1)$ and $(5, 0)$, where n refers to the number of real roots while ℓ refers to the number of pairs of complex roots. The cyclic quintics of minimum discriminant $\Delta = 14641 = 11^4$ found by Cohn are

$$V(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1, \quad \text{(Vandermonde's quintic)} \tag{1}$$

$$G(x) = x^5 + 2x^4 - 5x^3 - 2x^2 + 4x - 1. \tag{2}$$

Cohn manifested his surprise by the fact that his tables contained up to six distinct polynomials sharing the same discriminant and asked whether or not isodiscriminant polynomials would define the same number field. Subsequently, Hasse posed the same question, conjecturing the possible isomorphism between three quintics sharing a factor 47^2 in their discriminant.[13,14] The isomorphism conjectured by Hasse was confirmed by Zassenhaus and Liang,[15] who used a p-adic method to find explicit generating automorphisms of the Hilbert class field over $\mathbb{Q}(\sqrt{-47})$. Zassenhaus and Liang were apparently unaware of Cohn's earlier conjecture.

The field isomorphisms conjectured by Cohn were confirmed in 1974 by Cartier and Roy[16] who, using the same p-adic method of Zassenhaus and Liang,[15] reported tables containing explicit polynomial transformations interconnecting Cohn's isodiscriminant quintics for all three signatures. In particular, Cartier and Roy found the transformation

$$T(x) = x^4 - 4x^2 - x + 2, \tag{3}$$

to connect the roots of $V(x)$ to the roots of $G(x)$. The connection is as follows. Let $x_1, \ldots, x_5$ denote the roots of Vandermonde's celebrated quintic $V(x)$. Then, the roots of $G(x)$ are given by $T(x_1), \ldots, T(x_5)$. The inverse transformation, allowing the passage from the roots of $G(x)$ to the roots of $V(x)$ was not considered, neither by Zassenhaus and Liang nor by Cartier and Roy. While the intention of the earlier investigators was clearly to demonstrate the conjectured isomorphisms explicitly, there are a number of questions left open that need to be addressed in the context of the explicit determination of generating automorphisms of the Hilbert class field.

The specific representation of cyclic extensions of the rational field is important for a large class of physical problems related to the dynamics and stability properties of polynomial cycles. In this context, a complication is that, even for field extensions of relatively small degrees, to find a suitable representation of cyclic number fields is far from trivial. For instance, the literature contains a series of papers[17–19] dedicated to finding suitable forms to represent numbers in a cyclic quartic extension K of the rational field $\mathbb{Q}$. According to Hudson *et al.*,[20] a cyclic quartic extension K of $\mathbb{Q}$ may be expressed uniquely in the form

$$K = \mathbb{Q}\left(\sqrt{A(D + B\sqrt{D})}\right),$$

where A, B, C, D are integers such that (i) A is square free and odd, (ii) $D = B^2 + C^2$ is square free, $B, C > 0$, and (iii) the greatest common divisor of A and D is 1. No analogous result is known for quintic or higher order fields.

Before proceeding, we mention that the determination of number fields having the smallest discriminant has been a research topic since at least some 120 years. For, according to Mayer,[21] the absolute minimum discriminant for both signatures of cubics was determined in 1896 by Furtwängler.[22] Mayer himself obtained the minimum discriminants for the degree $n = 4$, discriminants also considered subsequently by Godwin[23–25] and others.[26,27] Early references for $n = 5$ include the work of Cohn[11] and the thesis by Hunter.[28] They both seem to be among the very first to use computers to investigate discriminants. More recent work on $n = 5$ was done by Pohst[29] and by Takeuchi.[30] Diaz y Diaz[31] reported a table containing 1077 totally real number fields of degree 5 having a discriminant less than 2 000 000. He finds two nonisomorphic fields of discriminant 1 810 969, a prime, and two nonisomorphic fields of discriminant $1\,891\,377 = 3^3 \cdot 70051$. All other number fields in his table are characterized by their discriminants. Among these fields, three are cyclic and four have a Galois closure whose Galois group is the dihedral group D_5. The Galois closure for all the other fields found has a Galois group isomorphic to the symmetric group S_5, meaning that the underlying quintics cannot be solved by radicals. Subsequently, Schwarz, Pohst and Diaz y Diaz[32] reported the determination of all algebraic number fields F of degree 5 and absolute discriminant less than 2×10^7 (totally real fields), respectively 5×10^6 (other signatures).

3. The New Poly-Transformations and their Actions

The polynomial $V(x)$, Eq. (1), represents an orbital equation of motion that is obtained for at least three paradigmatic physical models, in the so-called generating partition limit[33]: the quadratic map $x_{t+1} = 2 - x_t^2$, the Hénon map $(x, y) \mapsto (2 - x^2, y)$ and the canonical quartic map,[34,35] namely $x_{t+1} = (x_t^2 - 2)^2 - 2$. For details see, e.g. Refs. 36 and 37. The basic motivation for considering isomorphisms of Vandermonde's celebrated quintic, $V(x)$, is to see whether or not it is possible to interconnect distinct periodic orbits among themselves. While a general answer to this question does not seem easy, interesting extensions and generalizations were obtained that cast light into a promising theoretical framework to formulate such interconnections. This is what is reported here.

From a systematic search, we find the following transformations to provide *direct* passages from $V(x)$ to $G(x)$:

$$\begin{aligned}
D_1(x) &= -x^3 + x^2 + 3\,x - 2 = -(x-2)(x^2 + x - 1),\\
D_2(x) &= -x^3 + 2\,x = -x(x^2 - 2),\\
D_3(x) &= x^3 - x^2 - 2\,x + 1,\\
D_4(x) &= x^4 - 4\,x^2 - x + 2 = (x-2)(x+1)(x^2 + x - 1) = T(x),\\
D_5(x) &= -x^4 + x^3 + 4\,x^2 - 2\,x - 3.
\end{aligned}$$

Clearly, $D_4(x)$ coincides with $T(x)$, Eq. (3), previously found by Cartier and Roy. In addition, we find the following *inverse* automorphisms for the passage from $G(x)$

to $V(x)$:

$$\begin{aligned}
I_1(x) &= 4\,x^4 + 10\,x^3 - 15\,x^2 - 15\,x + 9,\\
I_2(x) &= x^4 + 2\,x^3 - 5\,x^2 - 3\,x + 3 = (x+1)(x^3 + x^2 - 6\,x + 3),\\
I_3(x) &= -2\,x^4 - 5\,x^3 + 7\,x^2 + 7\,x - 3 = -(x+1)(2\,x^3 + 3\,x^2 - 10\,x + 3),\\
I_4(x) &= -x^4 - 2\,x^3 + 5\,x^2 + 2\,x - 3,\\
I_5(x) &= -2\,x^4 - 5\,x^3 + 8\,x^2 + 9\,x - 5.
\end{aligned}$$

For every ℓ, $I_\ell(x)$ is the inverse of $D_\ell(x)$. Manifestly, these 10 transformations prove nonuniqueness of both sets. They significantly extend current knowledge concerning generating automorphisms.[38] The transformations are not all irreducible, and the degree of the inverses is always four.

Next, let the roots of these polynomials be so named that

$$\begin{aligned}
&V(x) : x_1 = -1.68, \quad x_2 = -0.83, \quad x_3 = 0.28, \quad x_4 = 1.30, \quad x_5 = 1.91,\\
&G(x) : y_1 = -3.22, \quad y_2 = -1.08, \quad y_3 = 0.37, \quad y_4 = 0.54, \quad y_5 = 1.39.
\end{aligned}$$

Applying $D_i(x)$ and $I_i(x)$ to the roots above produces their rearrangement as summarized in Tables 1 and 2. From Table 1 one sees that $y_5 = D_4(x_3)$, while from Table 2 one gets $x_3 = I_4(y_5)$, etc. In other words, the tables show unambiguously that $I_\ell(x)$ produces arrangements inverse to those produced by $D_\ell(x)$, for $\ell = 1, 2, 3, 4, 5$. In other words, the roots of $G(x)$ are invariant under the compositions $D_\ell(I_\ell(x))$, $\ell = 1, 2, 3, 4, 5$ while the roots of $V(x)$ are invariant under $I_\ell(D_\ell(x))$, $\ell = 1, 2, 3, 4, 5$. These compositions are of degrees 12 or 16, some of them can be

Table 1. Action of the direct transformations $D_\ell(x_i) \to y_j$.

	x_1	x_2	x_3	x_4	x_5
$D_1(x_j)$	y_4	y_1	y_2	y_5	y_3
$D_2(x_j)$	y_5	y_2	y_4	y_3	y_1
$D_3(x_j)$	y_1	y_5	y_3	y_2	y_4
$D_4(x_j)$	y_3	y_4	y_5	y_1	y_2
$D_5(x_j)$	y_2	y_3	y_1	y_4	y_5

Table 2. Action of the inverse transformations $I_\ell(y_i) \to x_j$.

	y_1	y_2	y_3	y_4	y_5
$I_1(y_j)$	x_2	x_3	x_5	x_1	x_4
$I_2(y_j)$	x_5	x_2	x_4	x_3	x_1
$I_3(y_j)$	x_1	x_4	x_3	x_5	x_2
$I_4(y_j)$	x_4	x_5	x_1	x_2	x_3
$I_5(y_j)$	x_3	x_1	x_2	x_4	x_5

factored over the rational integers. Their discriminants are usually given by quite large numbers and may contain large primes, for instance

$$\begin{aligned}D_5(I_5(x)) = {}& 16x^{16} + 160x^{15} + 344x^{14} - 1208x^{13} - 4631x^{12} + 2996x^{11}\\ & + 21362x^{10} - 2467x^9 - 49098x^8 + 1069x^7 + 60211x^6 - 6121x^5\\ & - 37383x^4 + 10540x^3 + 8777x^2 - 4797x + 643,\end{aligned}$$

whose discriminant is $2^{48} \cdot 19^4 \cdot 97 \cdot 103^4 \cdot 5014599729433540624461$. In contrast, one also finds pairs of almost identical factors:

$$\begin{aligned}D_4(I_4(x)) = {}& (x^4 + 2x^3 - 5x^2 - 2x + 2) \times (x^4 + 2x^3 - 5x^2 - 2x + 5)\\ & \times (x^8 + 4x^7 - 6x^6 - 24x^5 + 22x^4 + 30x^3 - 21x^2 - 10x + 5),\end{aligned}$$

whose discriminant is $-2^{20} \cdot 3^8 \cdot 5^{12} \cdot 61 \cdot 71 \cdot 1229 \cdot 7691$. Additional compositions reveal transformations of ever-growing degrees that produce root arrangements identical to the ones described and that display no obvious interconnections among them. Are $V(x)$ and $G(x)$ the only quintics sharing the minimum discriminant? This is what we investigate next.

The transformations $D_i(x)$ and $I_j(x)$ may be used to obtain infinite parametrized representations of irreducible isodiscriminant quintics. Explicitly, $D_i(x)$ produces the following set of n-parametrized transformations sharing the discriminant $11,^4$ independently of integer n:

$$\begin{aligned}P_n^{(1)}(x) = {}& x^5 - (5n + 8)x^4 + (10n^2 + 32n + 19)x^3\\ & - (10n^3 + 48n^2 + 57n + 4)x^2 + (5n^4 + 32n^3 + 57n^2 + 8n - 32)x\\ & - n^5 - 8n^4 - 19n^3 - 4n^2 + 32n + 23,\\ P_n^{(2)}(x) = {}& x^5 - (5n - 2)x^4 + (10n^2 - 8n - 5)x^3 - (0n^3 - 12n^2 - 15n + 2)x^2\\ & + (5n^4 - 8n^3 - 15n^2 + 4n + 4)x - n^5 + 2n^4 + 5n^3 - 2n^2 - 4n - 1,\\ P_n^{(3)}(x) = {}& x^5 - (5n - 7)x^4 + (10n^2 - 28n + 13)x^3 - (10n^3 - 42n^2 + 39n - 5)x^2\\ & + (5n^4 - 28n^3 + 39n^2 - 10n - 2)x - n^5 + 7n^4 - 13n^3 + 5n^2 + 2n - 1,\\ P_n^{(4)}(x) = {}& x^5 - (5n - 12)x^4 + (10n^2 - 48n + 51)x^3 - (10n^3 - 72n^2 + 153n - 96)x^2\\ & + (5n^4 - 48n^3 + 153n^2 - 192n + 80)x - n^5 + 12n^4 - 51n^3 + 96n^2\\ & - 80n + 23,\\ P_n^{(5)}(x) = {}& x^5 - (5n + 13)x^4 + (10n^2 + 52n + 61)x^3 + (10n^3 + 78n^2 + 183n + 119)x^2\\ & + (5n^4 + 52n^3 + 183n^2 + 238n + 70)x - n^5 - 13n^4 - 61n^3 - 119n^2\\ & - 70n + 23.\end{aligned}$$

These polynomials are not independent due to linear shifts induced by $D_i(x)$:

$$P_n^{(1)} = P_{n+2}^{(2)}, \quad P_n^{(2)} = P_{n+1}^{(3)}, \quad P_n^{(3)} = P_{n+1}^{(4)}, \quad P_n^{(4)} = P_{n-5}^{(5)}, \quad P_n^{(5)} = P_{n+1}^{(1)}.$$

They produce the same sequence of isodiscriminant cyclic quintics. In particular,

$$G(x) = P_{-2}^{(1)}(x) = P_0^{(2)}(x) = P_1^{(3)}(x) = P_2^{(4)}(x) = P_{-3}^{(5)}(x).$$

Similarly, the transformations $I_j(x)$ lead to irreducible quintics with discriminant 11^4:

$$\begin{aligned}
P_n^{(6)}(x) &= x^5-(5n-44)x^4+(10n^2-176n+770)x^3\\
&\quad -(10n^3-264n^2+2310n-6699)x^2\\
&\quad +(5n^4-176n^3+2310n^2-13398n+28974)x-n^5+44n^4-770n^3\\
&\quad +6699n^2-28974n+49841,\\
P_n^{(7)}(x) &= x^5-(5n-14)x^4+(10n^2-56n+74)x^3-(10n^3-84n^2+222n-183)x^2\\
&\quad +(5n^4-56n^3+222n^2-366n+210)x-n^5+14n^4-74n^3\\
&\quad +183n^2-210n+89,\\
P_n^{(8)}(x) &= x^5-(5n+16)x^4+(10n^2+64n+98)x^3-(10n^3+96n^2+294n+285)x^2\\
&\quad +(5n^4+64n^3+294n^2+570n+390)x-n^5-16n^4-98n^3\\
&\quad -285n^2-390n-199,\\
P_n^{(9)}(x) &= x^5-(5n+16)x^4+(10n^2+64n+98)x^3-(10n^3+96n^2+294n+285)x^2\\
&\quad +(5n^4+64n^3+294n^2+570,n+390)x-n^5-16n^4-98n^3\\
&\quad -285n^2-390n-199,\\
P_n^{(10)}(x) &= x^5-(5n+26)x^4+(10n^2+104n+266)x^3\\
&\quad -(10n^3+156n^2+798n+1337)x^2\\
&\quad +(5n^4+104n^3+798n^2+2674n+3298)x-n^5-26n^4-266n^3\\
&\quad -1337n^2-3298n-3191.
\end{aligned}$$

Clearly, despite the fact that $I_3(x) \neq I_4(x)$, both transformations produce identical parametrized forms: $P_n^{(9)}(x) = P_n^{(8)}(x)$. Analogously as before:

$$P_n^{(6)} = P_{n-6}^{(7)}, \quad P_n^{(7)} = P_{n-6}^{(9)}, \quad P_n^{(8)} = P_{n+12}^{(6)}, \quad P_n^{(9)} = P_{n-2}^{(10)}, \quad P_n^{(10)} = P_{n+2}^{(8)},$$

$$V(x) = P_{-3}^{(9)}(x) = P_{-5}^{(10)}(x) = P_{-3}^{(8)}(x) = P_9^{(6)}(x) = P_3^{(7)}(x).$$

The shifts above imply cyclic properties of the determinants defining discriminants.

4. Infinite Chains of Discriminant-Preserving Transformations

There is a richer way of generating infinite families of interrelated but not trivially connected polynomials sharing the same discriminant, minimal or not. Observing that $D_4(x) = T(x) = (x^2-2)^2 - 2 - x$ is a composition of a quadratic function, we are led to introduce families of recursive algorithms to generate unbounded sequences of discriminant-preserving polynomials, quintic or not.

Let $t_0(u) = u^2 - \alpha_0$ be a polynomial on an arbitrary variable u, and containing an arbitrary parameter α_0. With $t_0(u)$, build auxiliary polynomials $\{t_i(u)\}$

$$t_i(u) = t_{i-1}^2 - \alpha_i, \quad \text{for } i = 1, 2, \ldots, \tag{4}$$

where the several α_i are also chosen arbitrarily. Manifestly, the number of individual sets $\{t_i(u)\}$ is infinite. As an *ad hoc* example, we consider the specific sequence obtained by fixing $\alpha_i = 2$ for all i. In this case, the first few auxiliary polynomials are:

$$\begin{aligned}
t_0(u) &= u^2 - 2,\\
t_1(u) &= u^4 - 4u^2 + 2,\\
t_2(u) &= u^8 - 8u^6 + 20u^4 - 16u^2 + 2,\\
t_3(u) &= u^{16} - 16u^{14} + 104u^{12} - 352u^{10} + 660u^8 - 672u^6 + 336u^4 - 64u^2 + 2.
\end{aligned}$$

These polynomials are then used to define a chain of transformations, namely

$$T_i(u) = t_i(u) - u, \quad \text{for } i = 0, 1, 2, \ldots. \tag{5}$$

By construction, $T_1(u)$ coincides with $T(x)$ in Eq. (3), the transformation found by Cartier and Roy. The transformations $T_i(u)$ preserve discriminants, minimal or not, for quintics of any signature, cyclic or not. Generalized chains may be obtained analogously by iterating more complicated functions and by allowing α_i to vary as the iteration proceeds.

When applied to $V(x)$ and $G(x)$, the transformations $T_i(u)$ produce parametrized families which split dichotomically into either periodic or nonperiodic sequences of irreducible equivalent quintics. For instance, representing by $T_i(V)$ the operation of applying the transformation $T_i(x)$ to the roots of $V(x)$, we obtain a period-five polynomial cycle interconnecting three polynomials which repeat mod 5 indefinitely in the following order:

$$\begin{aligned}
&A(x) = T_0(V), \quad G(x) = T_1(V), \quad G(x) = T_2(V), \quad A(x) = T_3(V), \quad B(x) = T_4(V),\\
&A(x) = T_5(V), \quad G(x) = T_6(V), \quad G(x) = T_7(V), \quad A(x) = T_8(V), \quad B(x) = T_9(V),
\end{aligned}$$

where $G(x)$ is defined in Eq. (2) and

$$\begin{aligned}
A(x) &= x^5 + 2x^4 - 5x^3 - 13x^2 - 7x - 1, & \Delta_A &= 11^4,\\
B(x) &= x^5 + 2x^4 - 16x^3 - 24x^2 + 48x + 32, & \Delta_B &= 11^4 \cdot 2^{20}.
\end{aligned}$$

Clearly, $V(x)$ is not part of the above 5-cycle but leads to it. It is a sort of preperiodic equation of motion, mimicking the known behavior of preperiodic orbital points.[36] The above cycling of polynomials implies the existence of an infinity of additional direct transformations of ever increasing degrees allowing the passage from $V(x)$ to $G(x)$. In sharp contrast, the analogous sequence obtained from $T_i(G)$, for $i = 0, 1, 2, \ldots$, produces a nonrepeating sequence of quintics.

Many other never-repeating sequences of isomorphic irreducible quintics sharing similar discriminants may be extracted from additional parametrized families of totally real cyclic quintics. Four examples are

$$\begin{aligned}
\mathbb{A}_n(x) &= x^5 + (5n-9)x^4 + (10n^2 - 36n + 28)x^3 + (10n^3 - 54n^2 + 84n - 35)x^2 \\
&\quad + (5n^4 - 36n^3 + 84n^2 - 70n + 15)x + n^5 - 9n^4 + 28n^3 - 35n^2 + 15n - 1, \\
\mathbb{B}_n(x) &= x^5 - (5n+3)x^4 + (10n^2 + 12n - 3)x^3 - (10n^3 + 18n^2 - 9n - 4)x^2 \\
&\quad + (5n^4 + 12n^3 - 9n^2 - 8n + 1)x - n^5 - 3n^4 + 3n^3 + 4n^2 - n - 1, \\
\mathbb{C}_n(x) &= x^5 - (5n+15)x^4 + (10n^2 + 60n + 35)x^3 - (10n^3 + 90n^2 + 105n + 28)x^2 \\
&\quad + (5n^4 + 60n^3 + 105n^2 + 56n + 9)x - n^5 - 15n^4 - 35n^3 - 28n^2 - 9n - 1, \\
\mathbb{D}_n(x) &= x^5 - (5n+18)x^4 + (10n^2 + 72n + 35)x^3 - (10n^3 + 108n^2 + 105n + 16)x^2 \\
&\quad + (5n^4 + 72n^3 + 105n^2 + 32n - 2)x - n^5 - 18n^4 - 35n^3 - 16n^2 + 2n + 1.
\end{aligned}$$

The discriminant of these quintics does not depend on n, being 11^4 for $\mathbb{A}_n(x)$, $\mathbb{B}_n(x)$, and $\mathbb{C}_n(x)$, and 11^8 for $\mathbb{D}_n(x)$. Families of isodiscriminant quintics parametrized by more than one parameter can also be generated with $T_i(u)$ but this will not be pursued here.

To conclude this section, we observe that the above parametric forms share properties similar to a celebrated parametrized family of quintics, found by Emma Lehmer[39,40] to provide connections between the so-called Gaussian period equations and cyclic units. For instance, in the notation of Butler and McKay,[41] their common Galois group is 5T1. It is tempting to conjecture that the parametric forms reported here might also be linear combinations of Gaussian periods, providing interconnections between Gaussian period equations and cyclic units. This, however, remains to be ascertained.

5. Conclusions and Outlook

This paper reported direct and inverse transformations showing that in addition to a transformation found by Cartier and Roy, Cohn's conjectured isomorphism among minimum discriminant cyclic quintics can be established by nine new transformations, Therefore, altogether there are five direct and five inverse transformations which, when combined, reveal how orbital points are rearranged cyclically under the transformations.

An infinite chain of transformations was introduced and used to show that, apart from Cohn's pair of quintics, there is an apparently unbounded quantity of isomorphic quintics sharing the same field discriminant and group properties. The chain of transformations is generated by a simple recurrence relation, Eq. (5), and is valid to transform arbitrary equations of motion, of any degree, for systems governed by discrete maps. The chain introduced here may be naturally extended by replacing the quadratic functions underlying Eq. (5) by any arbitrary set of functions. In other words, the chain of transformations provide an effective tool to study the transformation properties of nonlinear equations of mathematical physics, so popular nowadays in practical applications. We hope to return to this in the future.

To conclude, we remark that the aforementioned question raised by Hasse was answered only in part by Zassenhaus and Liang. It remains to be determined whether or not there are additional analogous transformations interconnecting Hasse's triplet of quintics, which are not cyclic and have complex roots. Explicit expressions for generating automorphisms play a significant role in the study of Galois-theoretic aspects of iterated maps and Abelian groups.[42–45]

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