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Field discriminants of cyclotomic period equations

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We show that several orbital equations and orbital clusters of the quadratic (logistic) map coincide surprisingly with cyclotomic *period equations*, polynomials whose roots are Gaussian periods. An analytical expression for the field discriminant of period equations is obtained and applied to discover and to fill gaps in number field databases constructed by numerical search processes. Such expression allows easy assess to inessential divisors of conventional discriminants and sheds light into why numerical construction of databases is a hard problem. It also provides significant information about the organization of periodic orbits of the quadratic map.

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1. Introduction

It was recently discovered that, in the partition generating limit of the celebrated quadratic (logistic) map $x_{t+1} = 2 - x_t^2$, the familiar pair of period-3 equations of motion for trajectories in phase space can be extracted in a surprisingly double manner from either one of the following *orbital carriers*[1]:

$$\varphi_1(x) = \varphi_1(x;\sigma) = x^3 - \sigma\, x^2 - (\sigma^2 - 2\sigma + 3)\, x + \sigma^3 - 2\sigma^2 + 3\sigma - 1,$$
$$\varphi_2(x) = \varphi_2(x;\sigma) = x^3 - (1-\sigma)\, x^2 - (\sigma^2 + 2)\, x - \sigma^3 + \sigma^2 - 2\sigma + 1.$$

From them, choosing either $\sigma = 0$ or $\sigma = 1$, one obtains the following period-3 orbits:

$$\Phi(x) = \varphi_1(x;0) = \varphi_2(x;1) = x^3 - 3x - 1, \quad \Delta_{\Phi(x)} = 9^2, \tag{1}$$

$$\overline{\Phi}(x) = \varphi_1(x;1) = \varphi_2(x;0) = x^3 - x^2 - 2x + 1, \quad \Delta_{\overline{\Phi}(x)} = 7^2. \qquad (2)$$

Clearly, such observable *macroscopic* orbits of the quadratic map are obtained as independent "projections" from degenerate nonobservable *microscopic* carriers, either $\varphi_1(x)$ or $\varphi_2(x)$. Thus, rather than independent orbits, $\Phi(x)$ and $\overline{\Phi}(x)$ are seen to emerge as a dual pair of conjugated orbits. So, in classical dynamics, multiple microscopic carriers define a structural framework, or skeleton, for the orbits observed macroscopically in phase space, a mind-boggling fact. For details, see Ref. 1.

While working to determine orbital carriers for period-5 motions, we realized that the orbit $\overline{\Phi}(x)$ in Eq. (2) is one of the so-called *period equations* (defined in Sec. 2), which are the key players introduced in 1801 by Gauss to solve algebraically (i.e. by finite chains of radicals) the *cyclotomic equations*.[2–5] The orbit $\overline{\Phi}(x)$ has the lowest possible discriminant for cyclic cubics, namely $\Delta_{\overline{\Phi}(x)} = 7^2$. Its dual orbit, $\Phi(x)$, has the second smallest discriminant, viz. $\Delta_{\Phi(x)} = 9^2$. These remarkable facts sparked our interest in period equations. After computing many such equations, we found that they indeed coincide with several orbital equations and clusters of periodic trajectories of the quadratic map. This unexpected coincidence makes period equations an object of interest in the investigation of the dynamics of classical systems.

The purpose of this paper is to present a number of properties discovered for families of period equations. A key finding explored here in several applications boils down to Eq. (9), an exact expression for the field discriminant of period equations. Such expression reflects the fact, observed in our extensive computations, that field discriminants of period equations were found to be invariably given by powers of single prime numbers, associated in a simple manner with the equations.

Our motivation is related to applications in physics and, accordingly, we study equations with transitive cyclic groups, generically labeled as mT1, $m > 1$. Details concerning such applications may be found, e.g. in Refs. 1 and 6 and references therein. An additional long-term and more ambitious goal is to search for a systematic way to interconnect arithmetical families of equations of motion involved in period-doubling cascades of the map, forming towers of equations-within-equations of ever growing degrees.

In the remainder, we start by presenting briefly the context and a few concepts needed to understand and to define Eq. (9). Then, we show Eq. (9) to be an expeditious tool for the computation of the so-called *inessential discriminant divisors*,[7,8] factors of the conventional discriminant of equations of motion. Finally, we present tables and several explicit expressions for period equations and periodic orbits for relatively high periods of the quadratic map. Our tables provide a good feeling for the distribution and growth of the corresponding field discriminants. To start, one first needs to come to grips with period equations.

2. What are Period Equations?

Period equations are auxiliary polynomials with integer coefficients introduced by Gauss in his book *Disquisitiones Arithmeticæ*,[2] in the quest to solve cyclotomic equations with (*finite* chains of) radicals, i.e. to solve equations for the division of the perimeter of a circle into a given number of equal parts.[3–5] Since the methodology to obtain period equations is well described by Gauss, it has been thought unnecessary to give the arithmetical details in full but just a brief summary. A detailed textbook exposition is given in a classic book by Bachmann.[3] An enthusiastic opinion and summary of Bachmann's book is given by Dedekind.[9]

Consider the set Ω of the $p-1$ complex roots of the equation $x^p - 1 = 0$ where, here and throughout the paper, p is a prime number. To obtain Gauss' period equations one starts by first distributing the $p-1$ roots in Ω, into certain sums called "periods," as demonstrated in the *Disquisitiones Arithmeticæ*,[2] in paper 343: *Omnes radices* Ω *in certas classes (periodos) distribuuntur*. For two numbers e and f such that $p-1 = ef$, Gauss partitions all roots in Ω into e disjoint classes, thereby forming e periods η_i, each one consisting of a sum of f roots.

Let r be any complex root in Ω, for example $r = e^{2\pi i/p}$, and g a primitive root modulo p. Then, the "periods" (not the period equations) are given by the sums

$$\eta_i = \sum_{k=0}^{f-1} r^{g^{ke+i}}, \quad i = 0, 1, \ldots, e-1, \tag{3}$$

or, more explicitly, e sums of f distinct complex roots suitably selected from Ω

$$\begin{aligned}
\eta_0 &= r + r^{g^e} + r^{g^{2e}} + \cdots + r^{g^{(f-1)e}},\\
\eta_1 &= r^g + r^{g^{e+1}} + r^{g^{2e+1}} + \cdots + r^{g^{(f-1)e+1}},\\
&\vdots\\
\eta_{e-1} &= r^{g^{e-1}} + r^{g^{2me1}} + r^{g^{3e-1}} + \cdots + r^{g^{fe-1}}.
\end{aligned}$$

The *period equation* $\psi_e(x)$ of degree e associated with $p = ef + 1$ is defined as[2,3]

$$\psi_e(x) = \prod_{k=0}^{e-1} (x - \eta_k) = x^e + x^{e-1} + \alpha_2 x^{e-2} + \cdots + \alpha_e, \tag{4}$$

where all coefficients α_ℓ turn out to be integers (from $\mathbb{Z}$). Clearly, by construction, the roots of period equations are simply the "periods" η_i of Gauss.

To get a feeling for the characteristic distribution of $p-1$ into products ef, Table 1 illustrates the abundance and fast growth of the number of roots in η_i. For instance, the periods corresponding to the 50th prime of the form $p - 1 = 2f$ contain $(233-1)/2 = 116$ roots in the summation, the 500th prime contains $(3581-1)/2 = 1790$ roots, and the 10 000th prime contains $(104\,743-1)/2 = 52\,371$ roots. In contrast, for $e = 23$, the corresponding number of roots is, respectively, 380, 5096 and 132 020, an increase by factors of the order of 2.5.

Table 1. Primes $p = ef + 1$ as a function of e, defining the degree of the period equations. Primes are far more distant from each other for prime e because they have just a single signature (see text).

e	50th	200th	500th	1000th	2000th	3000th	5000th	10 000th
2	233	1229	3581	7927	17 393	48 619	48 619	104 743
3	577	2803	8089	17 539	38 011	104 953	104 953	225 217
4	577	2797	8009	17 657	38 153	105 269	105 269	225 217
5	1231	6151	17 551	38 231	82 421	225 781	225 781	479 821
6	577	2803	8089	17 539	38 011	104 953	104 953	225 217
7	2143	9829	28 057	59 627	127 709	351 121	351 121	748 987
8	1321	6521	17 761	38 609	82 793	226 241	226 241	482 441
9	2089	9829	27 361	59 833	128 467	351 361	351 361	747 991
10	1231	6151	17 551	38 231	82 421	225 781	225 781	479 821
11	3433	17 491	49 369	105 733	225 611	610 721	610 721	1 301 851
12	1321	6529	17 989	38 569	83 101	226 453	226 453	481 909
13	4759	21 191	59 359	128 519	275 549	746 227	746 227	1 592 683
14	2143	9829	28 057	59 627	127 709	351 121	351 121	748 987
15	2971	14 221	38 671	83 311	177 601	481 651	481 651	1 020 961
16	3041	14 081	38 321	82 913	177 409	481 697	481 697	1 024 337
17	6257	28 867	80 173	177 379	376 687	1 024 523	1 024 523	2 157 709
18	2089	9829	27 361	59 833	128 467	351 361	351 361	747 991
19	7867	34 961	95 153	203 339	434 303	1 164 511	1 164 511	2 460 881
20	3041	14 081	39 041	83 621	176 461	481 181	481 181	1 025 161
21	4789	22 051	60 271	128 941	277 747	751 549	751 549	1 588 819
22	3433	17 491	49 369	105 733	225 611	610 721	610 721	1 301 851
23	8741	45 403	117 209	249 229	533 831	1 443 389	1 443 389	3 036 461

3. The Dynamics Behind Period Equations

With hindsight, it is not difficult to recognize now that wide classes of polynomial equations of motion generated by a discrete-time physical model coincide with equations considered by Gauss and by Abel in the early 19th century.

Gauss introduced the systematic procedure reproduced in Sec. 2 and used it to investigate the subgroups of the group, subsequently named *Galois group*, of the cyclotomic equations. He found an explicit algorithm to solve a family of polynomials. However, although interesting for many reasons, cyclotomic equations form a relatively restricted class of polynomials.

Abel discovered that cyclotomic equations are nothing else than just a particular case of a much wider class of equations.[10] If the roots of an equation of arbitrary degree are connected in such a way that *all roots* may be rationally expressed as a function of one of them, say x and, designating by $\theta(x)$ and $\theta_1(x)$ any two other roots, where $\theta(x)$ and $\theta_1(x)$ are suitable rational functions, one finds that they obey (the commutative composition law)

$$\theta(\theta_1(x)) = \theta_1(\theta(x)), \tag{5}$$

then the equation in question will always be algebraically solvable by finite chains of radicals. Such general equations, called *Abelian equations* after Kronecker, are equations with n roots $x_1, x_2, \ldots, x_n$ which satisfy the relations

$$x_2 = \theta(x_1), \quad x_3 = \theta(x_2), \ldots, x_n = \theta(x_{n-1}), \quad x_1 = \theta(x_n), \tag{6}$$

where $\theta(x)$ is a rational function of x. As observed by Kronecker,[11] these general Abelian equations are essentially cyclotomic equations, closing the cycle.

Anyone familiar with equations of motions of discrete-time dynamical systems will immediately recognize that Eq. (6) corresponds to the discrete-time system

$$x_{t+1} = \theta(x_t). \tag{7}$$

For algebraic functions $\theta(x)$, iterating this equation generates sequences of polynomials whose roots are orbital points of the physical system described by $\theta(x)$. In general, orbital points are obtained numerically, i.e. only in an approximate way. However, by studying equations of motion exactly the emphasis is shifted from approximate *orbital points* in phase space, to the study of exact analytical properties and interrelations between equations of motion.[1] Clearly, chaotic dynamics cannot be described by Abelian equations. In the simple example chosen here, the partition generating limit, all orbits are unstable. However, carriers are valid generically, for arbitrary values of a. See Eq. (11) in Ref. 1.

For applications in physics, it is of interest to mention that in algebraic number theory, it can be shown that every cyclotomic field is an Abelian extension of the rational numbers $\mathbb{Q}$. In this context, an important discovery is the so-called Kronecker–Weber theorem, stating that every finite Abelian extension of $\mathbb{Q}$ can be generated by roots of unity, i.e. Abelian extensions are contained within some cyclotomic field. Equivalently, every algebraic integer whose Galois group is Abelian can be expressed as a sum of roots of unity with rational coefficients. For details, see, e.g. Edwards.[12] The key difficulty for application of the theorem above is buried in the word "some": to find explicit algorithms providing effective bridges to implement the postulated interconnections between Abelian extensions and cyclotomic fields. After proving that wealth exists, it seems important to find ways to get to it.

4. Results

4.1. *Expression for the field discriminant of period equations*

Two known invariants of any polynomial are its conventional discriminant D and the discriminant Δ of the number field K underlying the polynomial.[7,8,13] More technically, let $p(x)$ be a monic irreducible polynomial in $\mathbb{Z}(x)$ (i.e. an irreducible polynomial over the integers with nonzero coefficient of highest degree equal to 1), and r a root of $p(x) \in \mathbb{C}$. In addition, let K be the number field $\mathbb{Q}(r)$ and $\mathcal{O}$ the ring of (algebraic) integers in K. Then, the invariants D and Δ are interconnected by the simple-looking relation[7,8,13]

$$D = k^2 \Delta, \tag{8}$$

for some $k \in \mathbb{Z}$, where D is the discriminant of r and Δ is the discriminant of $\mathcal{O}$. Again, the trouble lies in the word "some."

As pointed out by Vaughan,[14] "*while D can be found by straightforward (if tedious) computation, the value of k is quite another story. According to Cohn,*[13] *page 77, for example, to determine k, one would have to test a finite number (which may be very large) of elements of K to see if they are integral.*"

Surprisingly, period equations form a wide class of equations for which the computation of Δ and k presents no difficulties and can be done using the following wide-ranging result.

For any prime $p = ef + 1$, the field discriminant Δ_e of the period equation $\psi_e(x)$ in Eq. (4) is given by

$$\Delta_e = \begin{cases} -p^{e-1}, & \text{if } (e-1) \bmod 4 = 1 \quad \text{and} \quad f \bmod 2 = 1, \\ p^{e-1}, & \text{if otherwise.} \end{cases} \tag{9}$$

Equivalently, Eq. (9) may also be written as

$$\Delta_e = \begin{cases} (-1)^{n_P} p^{e-1}, & \text{if } (e-1) \bmod 4 = 1, \\ p^{e-1}, & \text{if otherwise,} \end{cases} \tag{10}$$

where n_P is the number of *pairs* of complex roots of $\psi_e(x)$.

The *signature*[8] of a polynomial is (n_R, n_P), sometimes written more economically as n_R, where n_R is the number of real roots of $\psi_e(x)$. In the literature, number field tables are normally ordered using the magnitude $|\Delta_e|$ instead of Δ_e. Surprisingly, in Eq. (10), the sign of Δ_e is found to depend explicitly on the nature, odd or even, of the total number of pairs of complex roots of $\psi_e(x)$. Therefore, we expect the determination of this sign to be a nontrivial theoretical problem.

Equations (9) and (10) are empirical expressions distilled by consolidating numerical evidence gathered by tabulating thousands of field discriminants for period equations for primes $p = ef + 1$, for f varying up to a few thousands when $e \le 10$, and for f varying up to a few hundreds when $11 \le e \le 60$. Beyond $e = 60$, computations become too sluggish and are not pursued further. Despite the vast literature on cyclotomic equations, we have not been able to locate Eqs. (9) and (10). They correctly reproduce all numerically computed discriminants for the total mass of data investigated.

For every prime $p = 6f + 1$, Lehmer and Lehmer[15] reported coefficients for $\psi_e(x)$ in terms of L and M in the quadratic partition $4p = L^2 + 27M^2$. They also reported an explicit formula for the conventional discriminant D of $\psi_e(x)$. Four explicit examples of $\psi_e(x)$ and discriminants were given. However, while their $\psi_e(x)$ and the magnitudes of the conventional discriminants are correct, we find the sign of all their discriminants to be incorrect. In any case, nowadays it seems considerably safer and much easier to generate $\psi_e(x)$ numerically than to use the quite long and intricate expressions provided for the coefficients of $\psi_e(x)$.

For primes $p = 8f + 1$, Lehmer[16] investigated the use of difference sets and a class of octic residues of p to obtain conditions for octic period equations which, according to her, "*are rather rare; there are only three such primes less than ten thousand,* *namely* $p = 73,\ 6361$ *and* 9001". No explicit period equations were reported for these primes, only the conventional discriminant D for $p = 73$, viz. $D = 2^{54}\cdot 3^4\cdot 73^7$. For these rare octics, we find

$$\psi_8^{(73)}(x) = x^8 + x^7 + 5\,x^6 - 17\,x^5 - 46\,x^4 - 136\,x^3 + 320\,x^2 + 512\,x + 4096,$$

$$k^2 = 2^{54}\cdot 3^4, \quad \Delta = 73^7,$$

$$\psi_8^{(6361)}(x) = x^8 + x^7 + 398\,x^6 + 41\,446\,x^5 - 250\,747\,x^4 + 16\,689\,725\,x^3 + 486\,181\,868\,x^2 - 5\,601\,819\,268\,x + 224\,934\,834\,784,$$

$$k^2 = 2^{90}\cdot 3^{12}\cdot 5^{14}\cdot 11^{12}, \quad \Delta = 6361^7,$$

$$\psi_8^{(9001)}(x) = x^8 + x^7 + 563\,x^6 - 42\,614\,x^5 - 556\,282\,x^4, - 28\,875\,030\,x^3 + 863\,797\,853\,x^2 + 13\,357\,557\,897\,x + 926\,791\,611\,419,$$

$$k^2 = 2^{32}\cdot 3^4\cdot 5^{14}\cdot 11^4\cdot 23^{12}\cdot 43^{12}, \quad \Delta = 9001^7.$$

They are the 3rd, 196th and 271th octics for primes of the form $p = 8f + 1$, respectively. The discriminants for $p = 73$ agree. Although the reference table for totally complex octics lists polynomials containing field discriminants up to 122 digits, $\psi_8^{(6361)}(x)$ and $\psi_8^{(9001)}(x)$, with discriminants of 27 and 28 digits, respectively, are not listed.

Equation (9) gives a handy criterion to sort out equations with either $k^2 = 1$ or $k^2 \neq 1$. Thus, knowledge of field discriminants allows one to extract inessential discriminant divisors through a simple division of two (possibly very large) integers.

By avoiding the need for factorizing very large numbers, Eq. (9) allows a very significant reduction of the computations required to assess the arithmetical scaffolding underlying period equations.

4.2. *A general expression for α_2*

For primes $p = 3f + 1$, a general period equation $\psi_3(x)$ solving the cyclotomic trisection problem was given in 1872 by Bachmann, on pages 210–213 and 224–230 of his classic book *Die Lehre von der Kreisteilung*,[3] used properties of the elementary symmetric functions η_ℓ, to derive a one-parameter cubic that we write as

$$\psi_3(x) = x^3 + x^2 - \frac{1}{3}(p-1)x + A. \tag{11}$$

An equivalent form having the same discriminants is $-\psi_3(-x)$. Subsequently, in 1879, Cayley[17] reported

$$\psi_3(x) = x^3 + x^2 - \frac{1}{3}(p-1)x + fg - h^2, \tag{12}$$

tabulating f, g and h for the 11 primes $p = 3f + 1$ below 100, namely 7, 13, 19, 31, 37, 43, 61, 67, 73, 79 and 97. In 1901, Burnside[18] showed that Eq. (12) "may be completely solved, without the use of tables of any kind, by a number of trials which is small in comparison with the prime considered." As an example, for $p = 1213$, with five trials he finds the correct solution $x^3 + x^2 - 404x + 669 = 0$.

For $p = 5f + 1$, the period equation is a three-parameter quintic[19]

$$\psi_5(x) = x^5 + x^4 - \frac{2}{5}(p-1)x^3 + Cx^2 + Bx + A. \tag{13}$$

In general, for e odd, we find the coefficient α_2 of the third largest power of x in $\psi_e(x)$, Eq. (4), to be an integer given by

$$\alpha_2 = -\frac{e-1}{2e}(p-1) = -\frac{1}{2}(e-1)f. \tag{14}$$

We also find this same coefficient to be valid for e even and signature $(e, 0)$. For $e = 2$, the third largest power of x in $\psi_e(x)$ is in fact the constant term of a quadratic. We have not been able to locate this general coefficient in the literature.

It would be interesting to explore the possibility of, say, following Bachmann, to use the elementary symmetric functions to obtain expressions for additional coefficients.

4.3. *Tables of period equations*

All results reported here were obtained with a special purpose MAPLE routine written to generate systematically large sequences of period equations $\psi_e(x)$ for primes $p = ef + 1$, with e arbitrary but fixed. In this endeavor, period equations tabulated in 1875 by Reuschle[4] were helpful to validate our routine. Reuschle would be certainly amazed, perhaps shocked, to see that every period equation recorded in his invaluable and influential work of 13 years[20] could now be reproduced in fractions of a second or just a few seconds. For instance, the first 20 period equations for $p = 20f + 1$ were generated in 3.7 s, for $p = 30f + 1$ in 8.1 s, for $p = 40f + 1$ in 17.2 s, for $p = 50f + 1$ in 29.3 s and for $p = 60f + 1$ in 83.9 s, running MAPLE 2014 on a modest and aging DELL XPS 13 Ubuntu notebook. These simple tests generated much more period equations than reported in Reuschle's book.

In continuation, we compare our results with the ones in the detailed number field database of Klüners and Malle,[21–23] taken to be our reference tables. Malle[24] presents an impressive table listing the first 15 million cyclic cubic fields, complete up to field discriminant 10^6. These works contain links to a number of additional papers and online tables. For applications in physics, we mention the tables of totally real number fields up to degree 10 computed and maintained by Voight.[25] Most tables are concerned with number fields of relatively low-degree. The database of Klüner and Malle has minimal polynomials for fields up to degree 19, degrees not available in tables known to us. Of course, online number field tables are not at all concerned with period equations and, accordingly, the majority of their polynomials are not period equations. In fact, a byproduct of our work is precisely to have identified an infinite family of polynomials responsible for producing discriminants with the simplest possible structure, namely powers of single prime numbers.

For larger values of e, Table 5 presents information that goes well beyond what is presently available for equations with cyclic group.

4.3.1. *Period equations for primes of the form* $p = 3f + 1$

The conventional polynomial discriminant of the cubic $\psi_3(x)$, Eq. (11), is

$$D_c = -27A^2 - (6p-2)A + \frac{1}{27}(4p-1)(p-1)^2.$$

For $k = 1$, Eq. (9) implies $D_c = p^2$, a quadratic in A which has a rational and an integer value of A as solutions. For $k \neq 1$, both solutions are quadratic numbers. Therefore, the constraint $D_c = p^2$ provides a simple an efficient algorithm to sort out period equations with and without inessential discriminant divisors. For a fixed value of e, by increasing f successively, it is possible to extract in a systematic way, without omissions, a list of all primes $p = ef + 1$. For example, the 1187th cubic is $x^3 + x^2 - 22\,569\,406\,x + 41\,261\,890\,201$, for $p = 67\,708\,219$. The 1631st cubic is $x^3 + x^2 - 47\,112\,146\,x + 124\,449\,351\,881$, for $p = 141\,336\,439$. The 2405th cubic is $x^3 + x^2 - 111\,367\,856\,x + 452\,326\,735\,601$, for $p = 334\,103\,569$. The (weak) growth of the number of cubics with $k = 1$ obeys a power law.

For cyclic fields, the reference table displays discriminants for 211 cubics, the last one being $2\,989\,441 = 7^2 \cdot 13^2 \cdot 19^2$. Among them, there are 66 discriminants of the form p^2, the four ones larger than 1000 being $1021, 1153, 1213$ and 1327, respectively the 82th, 93th, 96th and 105th primes $p = 3f + 1$. These 66 cases lead us to suspect that the corresponding cubics could be period equations, or isomorphic forms of period equations. Indeed, they are.

Table 2 records A and k for the first 105 primes $p = 3f + 1$, characterized by field discriminants p^2. The sign and magnitude of A vary sensibly with p. Highlighted primes are not listed in the reference table. The magnitude of A for all primes in the reference table is smaller than 1000. With two exceptions, all missing primes highlighted in Table 2 have $|A| > 1000$. It is important to stress that the reference tables are not concerned with period equations. They contain many discriminants, which factor into products of powers of several primes and have minimal polynomials with coefficients that exceed 1000 considerably.

4.3.2. *Period equations for primes* $p = 4f + 1$

For cyclotomic primes $p = 4f + 1$ there are two classes of 4T1 cyclic polynomials, characterized by signatures 4 or 0. Table 3 contains the first 80 period equations, independent of signatures. Highlighting is used to discriminate signatures.

For $p = 4f + 1$, the reference tables[21] list 238 polynomials of signature 4 and 198 of signature 0. The largest cyclotomic prime listed is $p = 769$ for signature 4, and $p = 269$ for signature 0. Minimal polynomials for the totally real signature agree with ours, modulo the trivial substitution $x \mapsto -x$ or isomorphisms. In contrast, for signature $(0, 2)$, the reference table misses primes 109, 149, 157, 173, 181 and 229.

For $p = 37$ and signature 0, the minimal polynomial in the reference table is

$$f(x) = x^4 + 2x^3 + 20x^2 + 19x + 7, \quad k^2 = 3^4, \quad \Delta_e = 37^3,$$

Table 2. Solution set for cubics with transitive group $3T1$, ordered by the value of p, for the first 105 primes $p = 3f + 1$. Their field discriminant is $\Delta = p^2$ and the minimal polynomial is $f(x) = x^3 + x^2 - \frac{p-1}{3}x + A$. The *inessential discriminant divisors* are defined by k^2. Solutions for highlighted primes are not in the reference tables.

#	p	A	k	#	p	A	k	#	p	A	k
1	7	−1	1	36	379	365	5	71	853	1011	3^2
2	13	1	1	37	397	−544	2^2	72	859	−509	11
3	19	−7	1	38	409	−515	5	73	877	1819	1
4	31	−8	2	39	421	−343	7	74	883	1439	7
5	37	11	1	40	433	−16	2^3	75	907	−739	11
6	43	8	2	41	439	−504	$2 \cdot 3$	76	919	−1872	$2 \cdot 3$
7	61	−9	3	42	457	−220	2^3	77	937	−2221	1
8	67	5	3	43	463	343	7	78	967	1361	3^2
9	73	−27	3	44	487	−505	7	79	991	−2349	3
10	79	41	1	45	499	536	$2 \cdot 3$	80	997	−480	$2^2 \cdot 3$
11	97	−79	1	46	523	−891	3	81	1009	−1719	3^2
12	103	−61	3	47	541	521	7	82	1021	416	$2^2 \cdot 3$
13	109	−4	2^2	48	547	−81	3^2	83	1033	1913	7
14	127	80	2	49	571	−719	7	84	1039	2155	5
15	139	103	1	50	577	171	3^2	85	1051	−2608	2
16	151	−123	3	51	601	512	2^3	86	1063	2441	1
17	157	64	2^2	52	607	−1169	1	87	1069	2336	2^2
18	163	−169	1	53	613	999	3	88	1087	−2335	7
19	181	−67	5	54	619	321	3^2	89	1093	−1012	$2^2 \cdot 3$
20	193	143	3	55	631	−1075	5	90	1117	2565	3
21	199	59	5	56	643	−1024	$2 \cdot 3$	91	1123	1331	11
22	211	−125	5	57	661	−1273	3	92	1129	−2927	1
23	223	−256	2	58	673	−997	7	93	1153	−427	13
24	229	−212	2^2	59	691	128	$2 \cdot 5$	94	1171	347	13
25	241	125	5	60	709	1313	1	95	1201	2491	7
26	271	261	3	61	727	1104	$2 \cdot 3$	96	1213	629	13
27	277	236	2^2	62	733	1276	2^2	97	1231	−1003	13
28	283	304	2	63	739	−520	$2 \cdot 5$	98	1237	1741	11
29	307	−216	$2 \cdot 3$	64	751	1057	7	99	1249	2313	3^2
30	313	371	1	65	757	729	3^2	100	1279	−2179	11
31	331	−49	7	66	769	−1481	5	101	1291	−3347	5
32	337	25	7	67	787	−991	3^2	102	1297	−1345	13
33	349	−517	1	68	811	1592	2	103	1303	−2799	3^2
34	367	435	3	69	823	61	11	104	1321	3327	3
35	373	−221	7	70	829	−307	11	105	1327	−344	$2 \cdot 7$

while our Table 3 has

$$g(x) = x^4 + x^3 + 5x^2 + 7x + 49, \quad k^2 = 3^2 \cdot 7^2, \quad \Delta_e = 37^3.$$

Using either polynomial interpolation[26] or systematic computer search,[27] the number fields underlying these polynomials may be shown to be isomorphic, with two sets of four transformations interconnecting the polynomials. The passage from $f(x) \mapsto g(x)$ is accomplished by any of the following direct transformations:

$$D_1 = \frac{1}{3}(-x^3 - 2x^2 - 18x - 14),$$

Table 3. Solution set for biquadratics with field discriminant $\Delta = p^3$ for primes $p = 4f + 1$, and minimal polynomial $x^4 + x^3 + Cx^2 + Bx + A$, highlighted by signature. The *inessential discriminant divisors* are defined by k^2. "Seq" and "Sig" refer to the sequential enumeration of primes and signature. Among the first 80 primes there are 37 of signature 4 and 43 of signature 0. Note that $C = -\frac{3}{8}(p-1)$ for totally real quartics.

Seq	p	C,B,A	Sig	k	Seq	p	C,B,A	Sig	k
1,1	5	1,1,1	0	1	41,18	433	−162,839,−1003	4	$2\cdot3^3$
2,2	13	2,−4,3	0	3	42,19	449	−168,−477,335	4	$2\cdot5^3$
3,1	17	−6,−1,1	4	2	43,20	457	−171,1114,−2044	4	2
4,3	29	4,20,23	0	7	44,24	461	58,−1066,4601	0	$5\cdot109$
5,4	37	5,7,49	0	$3\cdot7$	45,25	509	64,350,8993	0	$11\cdot97$
6,2	41	−15,18,−4	4	2	46,21	521	−195,−814,−116	4	$2\cdot5^3$
7,5	53	7,−43,47	0	13	47,26	541	68,1454,6921	0	$3\cdot5\cdot43$
8,6	61	8,42,117	0	$3\cdot13$	48,27	557	70,−1288,7439	0	$7\cdot127$
9,3	73	−27,−41,2	4	2^4	49,22	569	−213,818,−20	4	$2\cdot5^3$
10,4	89	−33,39,8	4	2^4	50,23	577	−216,−36,1296	4	$2^4\cdot3^3$
11,5	97	−36,91,−61	4	2	51,24	593	−222,−1816,−3968	4	2^4
12,7	101	13,19,361	0	$5\cdot19$	52,25	601	−225,263,1256	4	$2^4\cdot3^3$
13,8	109	14,−34,393	0	$3\cdot5\cdot7$	53,28	613	77,1341,10773	0	$3^2\cdot7\cdot19$
14,6	113	−42,−120,−64	4	2^4	54,26	617	−231,−1581,−2374	4	2^7
15,7	137	−51,−214,−236	4	2	55,27	641	−240,1883,−4169	4	2
16,9	149	19,−121,635	0	$5\cdot31$	56,29	653	82,1102,13537	0	$7\cdot11\cdot19$
17,10	157	20,−206,517	0	$3\cdot37$	57,30	661	83,2107,9427	0	$3\cdot163$
18,11	173	22,292,667	0	43	58,28	673	−252,−2061,−4293	4	$2\cdot3^3$
19,12	181	23,215,975	0	$3\cdot5\cdot13$	59,31	677	85,127,16129	0	$13\cdot127$
20,8	193	−72,−205,−49	4	$2\cdot3^3$	60,32	701	88,482,17117	0	$7\cdot13\cdot19$
21,13	197	25,37,1369	0	$7\cdot37$	61,33	709	89,−1285,14853	0	$3\cdot7^2\cdot11$
22,14	229	29,−415,933	0	$3\cdot19$	62,34	733	92,−2428,9927	0	$3\cdot61$
23,9	233	−87,335,−314	4	2^4	63,35	757	95,899,19407	0	$3\cdot7^2\cdot13$
24,10	241	−90,−497,−739	4	2	64,29	761	−285,−1950,−2500	4	$2\cdot5^3$
25,11	257	−96,−16,256	4	2^7	65,30	769	−288,2259,−4617	4	$2\cdot3^3$
26,15	269	34,454,1945	0	$5\cdot61$	66,36	773	97,1691,17933	0	$11\cdot163$
27,16	277	35,329,2427	0	$3\cdot7\cdot19$	67,37	797	100,−1046,20557	0	$13\cdot157$
28,12	281	−105,123,236	4	2^7	68,31	809	−303,354,2348	4	$2\cdot7^3$
29,17	293	37,641,1853	0	73	69,38	821	103,2617,16327	0	$7\cdot193$
30,13	313	−117,450,−324	4	$2\cdot3^3$	70,39	829	104,−2746,14025	0	$3\cdot5\cdot67$
31,18	317	40,−416,2827	0	$7\cdot67$	71,40	853	107,−2399,17923	0	$3^2\cdot193$
32,14	337	−126,316,104	4	2^7	72,32	857	−321,2946,−7636	4	2
33,19	349	44,240,4203	0	$3^2\cdot67$	73,41	877	110,3234,16317	0	$3\cdot7\cdot31$
34,15	353	−132,684,−928	4	2^4	74,33	881	−330,2588,−4904	4	2^7
35,20	373	47,−303,4527	0	$3^2\cdot73$	75,34	929	−348,−2845,−4997	4	$2\cdot5^3$
36,21	389	49,851,3773	0	$5\cdot7\cdot13$	76,35	937	−351,−2401,−2434	4	$2^4\cdot3^3$
37,22	397	50,−918,3069	0	$3\cdot97$	77,42	941	118,3470,19625	0	$5\cdot229$
38,16	401	−150,−25,625	4	$2\cdot5^3$	78,36	953	−357,1370,1396	4	$2\cdot7^3$
39,17	409	−153,−230,548	4	$2\cdot5^3$	79,37	977	−366,−3969,−11911	4	2
40,23	421	53,−763,4557	0	$3\cdot7\cdot31$	80,43	997	125,−3801,19017	0	$3\cdot13\cdot19$

$$D_2 = \frac{1}{3}(-x^3 - x^2 - 20x - 6),$$
$$D_3 = \frac{1}{3}(x^3 + x^2 + 17x + 3),$$
$$D_4 = \frac{1}{3}(x^3 + 2x^2 + 21x + 14),$$

while the inverses, from $g(x) \mapsto f(x)$, are

$$I_1 = \frac{1}{21}(-2x^3 + 5x^2 - 17x - 7),$$
$$I_2 = \frac{1}{21}(-x^3 - 8x^2 - 19x - 35),$$
$$I_3 = \frac{1}{21}(x^3 + 8x^2 + 19x + 14),$$
$$I_4 = \frac{1}{21}(2x^3 - 5x^2 + 17x - 14).$$

4.4. *How rare are period equations?*

Table 4 provides a measure of the relative abundance and distribution of period equations. The upper part of the table puts into perspective data from the reference database,[21] while the lower part presents some analogous results for period equations of larger degrees. The first column gives polynomial degrees, the second records signatures, the third gives the number N of polynomials listed in the reference tables, not necessarily period equations. In the fourth column, numbers in black inform the basis of field discriminants of period equations contained in the reference tables. For instance, among 181 polynomials of degree 10 and signature 10 in the reference table, one finds four period equations whose discriminants are $41^9, 61^9, 101^9$ and 181^9. Complementing the table, in boldface blue, we show bases for the next few period equations in each sequence.

The fifth column lists the number of digits for the field discriminant of the last polynomial listed in the reference database. Thus, for degree 10, the discriminant of the 181th polynomial with signature 10 (i.e. with 10 real roots) is a number with 48 digits, while for signature 0, the discriminant of the 79th polynomial contains 22 digits. As indicated in the rightmost column, such polynomial is the 13th in the list of signature 0. For signature 10, the corresponding number in the rightmost column is $[2 \cdot 10^5]$. Such number is used to indicate that it would take too much time and resources to establish the sequential order of the 181th polynomial. In such cases, numbers in brackets give an estimate of the size of a prime whose number of digits would be about 48. For instance, the number of digits of $(2 \times 10^5)^9$ is 48.

From the rightmost column of Table 4, one sees clearly that there is no short supply of period equations. In particular, it is totally unreasonable to expect any table to contain them all. From the number of digits listed in the fifth column, it

Table 4. Relative abundance of period equations as a function of the degree of their minimal polynomials and signature. N refers to the number of fields listed in the reference tables. Highlighted blue boldface numbers are missing in the reference tables. See text for description of remaining data.

Deg	Sig	N	Discriminant bases	#dig	Seq
10	10	181	41, 61, 101, 181, **241, 281, 401, 421, 461, 521, 541, 601**	48	$[2 \cdot 10^5]$
	0	79	−11, −31, −71, −131, −151, −191, **−211, −251, −271**	22	13
11	11	40	23, 67, 89, **199, 331, 353, 397, 419, 463, 617, . . .**, 920371	357	$[10^{36}]$
12	12	102	73, **97, 193, 241, 313, 337, 409, 433, 457, 577, 601, 673**	35	47
	0	91	13, 37, 61, **109, 157, 181, 229, 277, 349, 373, 397, 421**	70	$[2 \cdot 10^6]$
13	13	3	53, 79, **131, 157, 313, 443, 521, 547, 599, 677, 859, 911**	27	4
14	14	75	29, **113, 197, 281, 337, 421, 449, 617, 673, 701, 757, 953**	83	$[6 \cdot 10^6]$
	0	56	−43, **−71, −127, −211, −239, −379, −463, −491, −547**	59	517
15	15	70	31, 61, **151, 181, 211, 241, 271, 331, 421, 541, 571, 601**	83	7216
16	16	30	**97, 195, 257, 353, 449, 577, 641, 673, 769, 929, 1153**	47	77
	0	29	17, **113, 241, 337, 401, 433, 593, 881, 977, 1009, 1201**	38	7
17	17	4	103, 137, 239, **307, 409, 443, 613, 647, 919, 953, 1021**	40	4
18	18	95	37, 73, **109, 181, 397, 433, 541, 577, 613, 757, 829, 937**	91	$[6 \cdot 10^4]$
	0	61	−19, **−127, −163, −199, −271, −307, −379, −489**	114	$[10^6]$
19	19	6	191, 229, 419, 457, 571, **647, 761, 1103, 1217, 1483, 1559**	50	5
21	21		**43, 127, 211, 337, 379, 421, 463, 547, 631, 673, 757**		
23	23		**47, 139, 277, 461, 599, 691, 829, 967, 1013, 1151, 1289**		
25	25		**101, 151, 251, 401, 601, 701, 751, 1051, 1151, 1201**		
27	27		**109, 163, 271, 379, 433, 487, 541, 757, 811, 919, 1297**		
29	29		**59, 233, 349, 523, 929, 1103, 1277, 1451, 1567, 1741**		
33	33		**67, 199, 331, 397, 463, 661, 727, 859, 991, 1123, 1321**		
43	43		**173, 431, 947, 1033, 1291, 1549, 1721, 1979, 2237**		

becomes clear that an attempt to include, say, all discriminants for degree 15 would demand a list with no less than 7216 entries. At the same time, Table 4 draws attention to how incomplete existing tables still are, particularly for polynomials of larger degrees. For instance, for degree 17, to include all polynomials having discriminants with up to 40 digits, would require adding just one more polynomial, the one corresponding to base 307. For degree 19, the reference table is already complete for discriminants with up to 50 digits. The last five lines in Table 4 record some data for polynomials that we have not found in online tables of number fields.

4.5. *Beyond tabulated polynomials*

Table 5 lists representative pairs of period equations characterized by totally real fields for primes $p = ef + 1$ where $e = 21, 23, 25, 27, 29, 33$ and 39. Coefficients are ordered according to Eq. (4), namely for $e = 21$ the coefficients are $\alpha_{22} = 1$, $\alpha_{21} = 11$, $\alpha_{20} = -55$, etc.

The degrees of the period equations in Table 5 go well beyond what is presently available in the literature for totally real cyclic fields and emphasize the easiness of generating such families systematically. We computed sequences with varying numbers of period equations, up to $e = 100$. Obviously, such sequences are simply

Table 5. The first two period equations for primes $p = ef + 1$ with $e = 21, 23, 25, 27, 29, 33, 39$, and field discriminants $\Delta_e = p^{e-1}$. Highlighted are (ℓ_P, ℓ_k), the number of digits in D and Δ_e. The identity $D = \Delta_K$ means $k^2 = 1$ (see Eq. (8)). Large values of $D - \Delta_e$ imply large inessential discriminant divisors.

Δ_e	Coefficients
43^{20} (33,33)	1,1,−20,−19,171,153,−816,−680,2380,1820,−4368,−3003,5005,3003,−3432, −1716,1287,495,−220,−55,11,1
127^{20} (104,43)	1,1,−60,−133,1305,4493,−10801,−62425,−2588,380273,489841,−832624,−2307149, −540263,3165855,3188668,−157753,−1481414,−380716,205872,50035,−14459
47^{22} (37,37)	1,1,−22,−21,210,190,−1140,−969,3876,3060,−8568,−6188,12376, 8008,−11440,−6435,6435,3003,−2002,−715,286,66,−12,−1
139^{22} (126,48)	1,1,−66,−147,1630,5648,−16457,−92686,18441,709360,832638,−2239299,−5679764, 156443,12673530,11318727,−6468097,−14166332,−3186420,5386949,2918745, −436718,−516219,−63941
101^{24} (134,49)	1,1,−48,−43,946,752,−9993,−6962,62052,37341,−234195,−119366,538390,226505, −737819,−249907,571793,151052,−224456,−42136,35494,2561,−1633,−57,19,1
151^{24} (150,53)	1,1,−72,−161,1991,6935,−23789,−131523,59219,1219472,1274267,−5134575,−12138942, 4263646,39567816,30337248,−42180110,−75903945,−9872226,55689151, 38006340,−5737119,−14814116,−5029783,−190198,111103
109^{26} (159,53)	1,1,−52,−47,1128,914,−13369,−9612,95357,60102,−425693,−231576,1201391,553157, −2121177,−810403,2271851,706862,−1399735,−342875,461618,78149,−74294, −4948,4861,−271,−34,1
163^{26} (174,58)	1,1,−78,−175,2388,8354,−33013,−180016,127774,1968453,1782518,−10489594,−23282154, 17166283,103371060,63556949,−176991137,−284942103,20494295,355135692,239709074, −92044084,−184127684,−70412054,8685910,12922671,3203525,255583
59^{28} (50,50)	1,1,−28,−27,351,325,−2600,−2300,12650,10626,−42504,−33649,100947,74613,−170544, −116280,203490,125970,−167960,−92378,92378,43758,−31824,−12376,6188, 1820,−560,−105,15,1
233^{28} (297,67)	1,1,−112,−91,5198,3644,−132219,−83238,2053518,1187959,−20553532,−11071128, 136460842,69042962,−609473492,−292259011,1836592125,845018358,−3706016039, −1661552324,4906886664,2177019390,−4095369839,−1819962089,1998032360, 895362174,−490947342,−221892059,42927079,19524467
67^{32} (59,59)	1,1,−32,−31,465,435,−4060,−3654,23751,20475,−98280,−80730,296010,230230,−657800, −480700,1081575,735471,−1307504,−817190,1144066,646646,−705432,−352716,293930, 125970,−77520,−27132,11628,3060,−816,−136,17,1
199^{32} (250,74)	1,1,−96,−217,3795,13403,−74197,−394231,599821,6350170,2832807,−56803105, −104532088, 244229488,932758015,−5618002,−3890173018,−4529747891,6495127532,18110944809, 4574986912,−26694143816,−29645825157,6037403432,30417132332,15468969217, −6737165737,−8515927088,−1017730658,1409177433,395068072,−75602260, −21210003,2947097
79^{38} (73,73)	1,1,−38,−37,666,630,−7140,−6545,52360,46376,−278256,−237336,1107568,906192,−3365856, −2629575,7888725,5852925,−14307150,−10015005,20030010,13123110,−21474180,−13037895, 17383860,9657700,−10400600,−5200300,4457400,1961256,−1307504,−490314, 245157,74613,−26334,−5985,1330,190,−20,−1
157^{38} (310,84)	1,1,−76,−71,2556,2222,−50313,−40520,646279,479776,−5720417,−3892342,35931891, 22265255,−162617513,−91086546,533275855,267613697,−1265136580, −562372122,2154121978,835520674,−2594978102,−861831376,2164301236,603623323, −1211061590,−280758113,434291871,84841877,−93357668,−15992102,10935603, 1639529,−599706,−67036,11826,272,−49,1

too big to record here, although they provide significant insights concerning their organization, growth, as well as minimum discriminants of $\psi_e(x)$.

Note that for the larger degrees, discriminants contain increasingly larger number of digits, and become harder and harder to factor without better and dedicated resources.

4.6. *Orbits and orbital clusters of Pincherle's map* $x_{t+1} = 2 - x_t^2$

All period equations considered so far had discriminants given by powers of single primes. It is important to mention that n-periodic orbits of the quadratic map do not necessarily have equations of motion defined by n degree polynomials with integer coefficients as is the case for Eqs. (1) and (2). Most of the times, orbits appear as *orbital clusters* entangling arithmetically together several distinct orbits with the same period.

For instance, for period-4, in the partition generating limit, the limit studied as early as 1920 by Pincherle,[6,28] the three individual period-4 orbits emerge as one single orbit and a cluster formed by two orbits

$$\begin{aligned} o_{4,1}(x) &= x^4 + x^3 - 4\,x^2 - 4\,x + 1, \quad \Delta = 3^2 \cdot 5^3, \\ c_{4,1}(x) &= x^8 - x^7 - 7\,x^6 + 6\,x^5 + 15\,x^4 - 10\,x^3 - 10\,x^2 + 4\,x + 1, \quad \Delta = 17^7. \end{aligned}$$

The pair of orbits of the cluster decompose as $c_{4,1}(x) = o_{4,2}(x) \cdot o_{4,3}(x)$ where

$$\begin{aligned} o_{4,2}(x) &= x^4 - \frac{1}{2}(1+\sqrt{17})x^3 - \frac{1}{2}(3-\sqrt{17})x^2 - (2-\sqrt{17})x - 1, \quad \Delta = 17^2 - 68\sqrt{17}, \\ o_{4,3}(x) &= x^4 - \frac{1}{2}(1-\sqrt{17})x^3 - \frac{1}{2}(3+\sqrt{17})x^2 - (2+\sqrt{17})x - 1, \quad \Delta = 17^2 + 68\sqrt{17}. \end{aligned}$$

Remarkably, the single orbit $o_{4,1}(x)$ is not a period equation, while the cluster $c_{4,1}(x)$ is. For period-5, we obtain

$$\begin{aligned} o_{5,1}(x) &= x^5 - x^4 - 4\,x^3 + 3\,x^2 + 3\,x - 1, \quad \Delta = 11^4, \\ c_{5,1}(x) &= x^{10} + x^9 - 10\,x^8 - 10\,x^7 + 34\,x^6 + 34\,x^5 - 43\,x^4 - 43\,x^3 \\ &\quad + 12\,x^2 + 12\,x + 1, \quad \Delta = 3^5 \cdot 11^9, \\ c_{5,2}(x) &= x^{15} - x^{14} - 14\,x^{13} + 13\,x^{12} + 78\,x^{11} - 66\,x^{10} - 220\,x^9 + 165\,x^8 \\ &\quad + 330\,x^7 - 210\,x^6 - 252\,x^5 + 126\,x^4 + 84\,x^3 - 28\,x^2 - 8\,x + 1. \quad \Delta = 31^4. \end{aligned}$$

As before for period-4, $c_{5,1}(x) = o_{5,2}(x) \cdot o_{5,3}(x)$, where

$$\begin{aligned} o_{5,2}(x) &= x^5 + \frac{1}{2}(1+\sqrt{33})x^4 - x^3 - \frac{1}{2}(9+3\sqrt{33})x^2 - (6+\sqrt{33})x - 1, \quad \Delta = 11^2, \\ o_{5,3}(x) &= x^5 + \frac{1}{2}(1-\sqrt{33})x^4 - x^3 - \frac{1}{2}(9-3\sqrt{33})x^2 - (6-\sqrt{33})x - 1, \quad \Delta = 11^2. \end{aligned}$$

While $o_{5,1}(x)$, the celebrated quintic of Vandermonde,[27] defines a single orbit, $c_{5,1}(x)$ and $c_{5,2}(x)$ are entanglements of 2 and 3 period-5 orbits, respectively. Moreover,

$o_{5,1}(x)$, $c_{5,1}(x)$ and $c_{5,2}(x)$ have $k^2 = 1$, and $o_{5,1}(x)$ and $c_{5,2}(x)$ are period equations. As for period-4, individual orbits composing clusters have coefficients given by complicated algebraic numbers, not integers. Therefore, orbital clusters may also contain discriminants involving products of powers of multiple primes. It is quite challenging to decompose orbital clusters combining more than two orbits, particularly when they combine an odd number of orbits. However, the coefficients of such decompositions hide the secretest truth and most interesting relations among numbers which fix orbital individuality.

Analogously, we find the 18 orbits of period-7 to emerge in three clusters, with 3, 6 and 9 orbits

$$\begin{aligned}
c_{7,1}(x) &= x^{21} - x^{20} - 20\,x^{19} + 19\,x^{18} + 171\,x^{17} - 153\,x^{16} - 816\,x^{15} + 680\,x^{14} \\
&\quad + 2380\,x^{13} - 1820\,x^{12} - 4368\,x^{11} + 3003\,x^{10} + 5005\,x^{9} - 3003\,x^{8} \\
&\quad - 3432\,x^{7} + 1716\,x^{6} + 1287\,x^{5} - 495\,x^{4} - 220\,x^{3} + 55\,x^{2} + 11\,x - 1, \\
c_{7,2}(x) &= x^{42} + x^{41} - 42\,x^{40} - 42\,x^{39} + \cdots - 3267\,x^{4} - 3267\,x^{3} + 44\,x^{2} + 44\,x + 1, \\
c_{7,3}(x) &= x^{63} - x^{62} - 62\,x^{61} + 61\,x^{60} + \cdots + 40920\,x^{4} + 5456\,x^{3} - 496\,x^{2} - 32\,x + 1.
\end{aligned}$$

All three have $k^2 = 1$ and discriminants 43^{20}, $3^{21} \cdot 43^{41}$ and 127^{62}, respectively. Manifestly, only clusters $c_{7,1}(x)$ and $c_{7,3}(x)$ are period equations.

There are only quite a small number of nonarithmetically entangled orbits, meaning simply that, most of the times, the coefficients of periodic orbits will be given by more complicated algebraic numbers, not integers. For periods 9, 10, 11 and 12, the only periodic orbits with integer coefficients are

$$\begin{aligned}
o_{9,1}(x) &= x^{9} - x^{8} - 8\,x^{7} + 7\,x^{6} + 21\,x^{5} - 15\,x^{4} - 20\,x^{3} + 10\,x^{2} + 5\,x - 1, \quad \Delta = 19^{8}, \\
o_{9,2}(x) &= x^{9} - 9\,x^{7} + 27\,x^{5} - 30\,x^{3} + 9\,x - 1, \quad \Delta = 3^{22}, \\
o_{10,1}(x) &= x^{10} - 10\,x^{8} + 35\,x^{6} - x^{5} - 50\,x^{4} + 5\,x^{3} + 25\,x^{2} - 5\,x - 1, \quad \Delta = 5^{17}, \\
o_{11,1}(x) &= x^{11} - x^{10} - 10\,x^{9} + 9\,x^{8} + 36\,x^{7} - 28\,x^{6} - 56\,x^{5} + 35\,x^{4} \\
&\quad + 35\,x^{3} - 15\,x^{2} - 6\,x + 1, \quad \Delta = 23^{10}, \\
o_{12,1}(x) &= x^{12} + x^{11} - 12\,x^{10} - 11\,x^{9} + 54\,x^{8} + 43\,x^{7} - 113\,x^{6} \\
&\quad - 71\,x^{5} + 110\,x^{4} + 46\,x^{3} - 40\,x^{2} - 8\,x + 1, \quad \Delta = 5^{9} \cdot 7^{10}, \\
o_{12,2}(x) &= x^{12} - 12\,x^{10} + x^{9} + 54\,x^{8} - 9\,x^{7} - 112\,x^{6} \\
&\quad + 27\,x^{5} + 105\,x^{4} - 31\,x^{3} - 36\,x^{2} + 12\,x + 1, \quad \Delta = 3^{18} \cdot 5^{9}, \\
o_{12,2}(x) &= x^{12} + x^{11} - 12\,x^{10} - 12\,x^{9} + 53\,x^{8} + 53\,x^{7} \\
&\quad - 103\,x^{6} - 103\,x^{5} + 79\,x^{4} + 79\,x^{3} - 12\,x^{2} - 12\,x + 1, \quad \Delta = 3^{6} \cdot 13^{11}.
\end{aligned}$$

They all have $k^2 = 1$ and only $o_{9,1}(x)$ and $o_{11,1}(x)$ are period equations. The discriminants of $o_{9,1}(x)$ and $o_{9,2}(x)$ are the first and second smallest for cyclic equations of degree nine, while $o_{10,1}(x)$ has the third smallest and $o_{11,1}(x)$ the smallest possible discriminant for cyclic polynomials of degrees 10 and 11, respectively.[21] Of the 335 period-12 orbits, only the three above have integer coefficients. They are not period equations, but are the triplet of cyclic polynomials with minimum discriminants. The 630 period-13 orbits emerge as three rather big polynomials, of degrees 1365, 2730 and 4095, conglomerating 105, 210 and 315 orbits, respectively. The coefficients of the individual orbits must involve algebraic numbers with exquisite symmetry properties that would be interesting to study, despite the challenge of the task. For polynomial maps, an exact equation giving the total number of periodic orbits as a function of the period is given in Ref. 29.

Among orbits and orbital clusters of the quadratic map one finds the startling phenomenon of *period inheritance*.[31] A detailed discussion of orbits and clusters for the quadratic map will be presented elsewhere.

5. Conclusions and Outlook

Motivated by the remarkable fact that several periodic orbits and orbital clusters of the quadratic map coincide with period equations, this paper reported a number of properties of period equations uncovered by computing large sets of them, and consolidating trends observed.

It was found that period equations may be systematically generated and enumerated, with no omissions, for primes of the form $p = ef + 1$. This fact allows one to recognize and to fix some gaps in tables of number fields currently available in the literature. It also makes clear that, due to the abundance of period equations, there is no hope of ever producing "complete" tables. Fortunately, however, period equations are not difficult to generate when needed, using currently available computer algebra systems. Maybe future versions of such types of software will incorporate intrinsic functions for this purpose.

The design of an efficient routine for the systematic determination of classes of solutions ended up disclosing exact theoretical expressions, conjectures, which seem hard to come by theoretically and which are now ready to be challenged by traditional demonstrations. For instance, we found a simple and general closed-form expression, Eq. (9), for the field discriminant of cyclotomic period equations. As shown, such expression grants direct access to the so-called inessential discriminant divisors[8] buried in conventional polynomial discriminants and normally quite difficult to determine. Equation (9) provides an easy criterion to sort out equations with either $k^2 = 1$ or $k^2 \neq 1$, sets that we find to contain an unbounded quantity of equations and emerging intertwined with a quite irregular distribution of magnitudes and signs. Additional analytical results are reported in Sec. 4.2, in particular by Eq. (14), and in Secs. 4.3 and 4.3.1.

As is the case for $\varphi_1(x)$ and $\varphi_2(x)$, note that the *branch ambiguity* of the square root signs in $o_{4,2}(x)$ and $o_{4,3}(x)$, as well as in $o_{5,2}(x)$ and $o_{5,3}(x)$, make such orbits to be only *formally* well defined. In fact, to represent unambiguous orbits, such forms still depend on fixing the branch for the square root that they contain. By suitable branch choice, the formally ambiguous expression $o_{4,2}(x)$ may be "projected" into anyone of the two branch-fixed orbits. The same is valid for the ambiguous $o_{4,3}(x)$

that may also be selected to represent anyone of the two branch-fixed orbits. The existence of root-ambiguity before fixing branches is a simple arithmetical consequence of the multivaluedness of numbers in the roots.

The systematic generation of period equations allows one to enumerate unambiguously classes of number fields. In some sense, such enumeration resembles somewhat the arithmetic order discovered to exist among unordered binary labels associated with the symbolic dynamics of the quadratic map.[30] The identification of period equations as periodic orbits and clusters of the quadratic map lends hope that, eventually, it may be possible to disclose analytically the regular processes underlying the organization of bifurcation cascades observed so frequently in physical models. A promising application is to detect and classify orbital interdependencies in classical dynamics.[31] An open question is to understand why some orbital equations are *not* period equations, while some clusters are. A further enticing open question is to determine if, as for period equations, other discriminant regularities abundantly present in number field databases are associated with additional families of polynomials yet to be discovered.

Note added (October 31, 2019)

While searching for references to Eqs. (9) and (14), with the help of the internet and kind leads and feedback provided by Profs. G. E. Andrews, B. C. Berndt, R. J. Evans, K. Győry, F. Lemmermeyer, W. Narkiewicz, A. Schinzel, A. Ware, H. C. Williams and K. S. Williams, it was possible to uncover the following facts.

Prof. Evans pointed out that, up to sign, Eq. (9) was given by Neto *et al.*[32] By Galois theory, there is only one possible subfield K and, accordingly, we identify $[K : \mathbb{Q}] = e$. The field K is generated over $\mathbb{Q}$ by the e roots of the period equation $\psi_e(x)$ in Eq. (4). Our Eqs. (9) and (10) agree with the magnitude of Δ_e reported by Neto *et al.*, and, in addition, they provide the proper signs for all the cases.

Prof. Narkiewicz pointed out that our Eq. (9) is correct and follows from the conductor-discriminant formula, see, e.g. Theorem 3.11 in the book *Introduction to Cyclotomic Fields*.[34] He also mentions that Gurak[33] presented a procedure to compute the beginning coefficients of the minimal polynomials of the period equations. It was not verified if Gurak's results lead or not to our Eq. (14).

It was not yet possible to locate exact reference to Eq. (14) in the literature. However, using cyclotomic numbers and other results from the book *Gauss and Jacobi Sums*,[35] Prof. K. S. Williams kindly sent us a general proof that Eq. (14) is indeed correct, as well as an expression for the missing case of totally complex fields with signature $(0, e/2)$, namely

$$\alpha_2 = \frac{e + p - 1}{2e} = \frac{1}{2}(f + 1). \tag{15}$$

This coefficient matches exactly all our computational data. The author expresses his gratitude to all persons involved for their generous contributions.

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