

INVESTIGATING AN AUTOMATON OF "CLASS 4"

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We study the one-dimensional totalistic rule 20 with five cells per neighborhood. We find two transients both characterized by finite correlation lengths. The first transient of about 100 time-steps terminates in the glider phase. Collisions of gliders determine the second transient of about 30,000 time-steps. The final state has a finite magnetization and is indistinguishable from class 2 behaviour.

1. Introduction

Cellular automata^{1,2} are non-linear dynamical systems of many degrees of freedom that are defined by a discrete rule and which create patterns in space and time. For some rules these patterns are very simple (class 1 and 2) and for other rules (class 3) one finds self-similar patterns without characteristic length scale as one would expect in a chaotic system. Finally there are also some other rules which Wolfram³ grouped together as "class 4".

About "class 4" automata very little is concretely known. Wolfram proposed³ that automata in this class can be used as Turing machines, i.e. as universal computers. He found that these automata produce various types of stable structures (e.g. gliders or glider-guns) and illustrated the behaviour of this class on various examples, the most famous one being the "Game of Life".⁴

On the other hand it is not even clear that a "class 4" is well-defined. In particular there are intrinsic problems to decide to which class an automaton belongs.⁵ Not all automata that were originally believed to belong to this class have the same type of stable structures and it is in fact not easy to say what these automata have all in common.

The purpose of this paper is to study quantitatively one automaton that has been proposed to belong to class 4, in a similar way as rule 22 was extensively studied^{6,7} as a prototype for an automaton of class 3. In fact, we will investigate the simplest possible candidate, namely the one-dimensional totalistic rule 20 which Wolfram already discussed in Ref. 3. We consider this work to be a case study which

tells what could happen to a complex rule without claiming that the behaviour we find is necessarily generic to all the automata that have been said to belong to class 4.

2. Definition of the Automaton and Numerical Techniques

The automaton that we investigate in this paper is defined on a linear closed chain of N sites, each site i having a binary degree of freedom σ_i , i.e., $\sigma_i = 0$ or 1. The rule that defines the time evolution of a given site i depends on the nearest and next-nearest neighbors of i and on the value of i itself, i.e. it depends on the five variables $\sigma_{i-2} \dots \sigma_{i+2}$. It can be stated as follows:

$$\begin{aligned} \sigma_i(t+1) = 1 & \quad \text{if } \sigma_{i-2}(t) + \sigma_{i-1}(t) + \sigma_i(t) + \sigma_{i+1}(t) + \sigma_{i+2}(t) = 2 \text{ or } 4 \\ \sigma_i(t+1) = 0 & \quad \text{otherwise.} \end{aligned} \quad (1)$$

Since the value at the next time only depends on the sum of the old variables such a rule has been called "totalistic"¹ and is in fact the "rule 20" in Wolfram's classification of totalistic automata for neighborhoods of nearest and next-nearest sites.³

At $t = 0$ we start with a random configuration that has a fraction p of ones and a fraction $1 - p$ of zeroes. The case $p = \frac{1}{2}$ which corresponds to the usual case of random initial configuration is illustrated in Fig. 1. We see the time evolution over 250 time-steps as shown also by Wolfram.³ After a transient of about 100 time-steps most of the ones disappear and only some stable, periodic structures, called "gliders", survive. Wolfram has investigated periods and frequencies of occurrence of the different types of gliders that can be created. Some gliders are fixed in space while others move with a constant velocity. As long as a glider does not encounter sites with non-zero value it will be unaltered for all times.

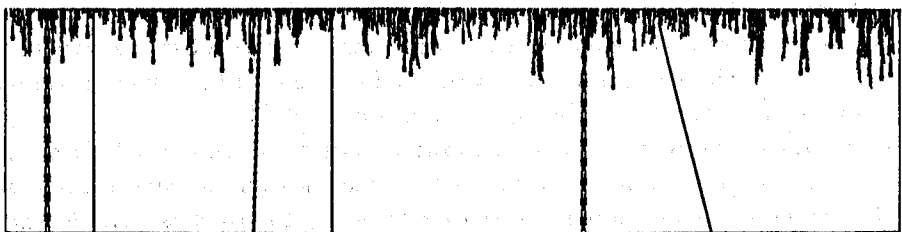


Fig. 1. Evolution of a random configuration ($p = \frac{1}{2}$) of $N = 3904$ sites over 250 time-steps. Ones are shown in black and zeroes in white.

It is the aim of this paper to investigate the transient and the glider phase quantitatively. For this purpose we will calculate the magnetization $m(t)$ and the damage $\Delta(t)$. The magnetization is just the fraction of ones at time t . At $t = 0$ one has $m(0) = p$. The damage is defined when one compares the evolution of two

configurations $\{\sigma\}$ and $\{\rho\}$ as the "Hamming distance" between the two configurations:

$$\Delta(t) = \frac{1}{2N} \sum_{i=1}^N |\sigma_i(t) - \rho_i(t)|. \quad (2)$$

In other words, the damage is the fraction of the sites on which the two configurations differ.

The questions we want to address are: How does the magnetization decay in the transients? Does the glider phase have a finite magnetization in the limit of $N \rightarrow \infty$? How does an infinitesimal initial damage spread out? The last question is the typical criterion for a chaotic behaviour: In a chaotic situation an infinitesimal perturbation is infinitely amplified.

We have simulated the automaton by using multi-spin-coding techniques^{8,9}: In each of the 64 bits of a computer word we store the binary variable of one site. Neighboring sites belong to different words. The rule is implemented through the following logical function

$$\sigma_i(t+1) = \neg \left[\left(\bigoplus_{k=i-2}^{i+2} \sigma_k(t) \right) \vee \left(\bigwedge_{k=i-2}^{i+2} \neg \sigma_k(t) \right) \right] \quad (3)$$

where \wedge , \vee , \oplus and \neg are the Boolean operations "and", "or", "exclusive or" and the complement. Using logical bit-by-bit operations, as implemented on the CRAY (and, or, xor, compl) or on the Convex (kiand, kior, kieor, knot), Eq. (2) is executed simultaneously for the 64 sites within one word.

The same kind of parallelism can also be implemented when calculating the damage by taking the "exclusive or" of the two configurations and Eq. (2) can then be calculated like the magnetization by using the command "popcnt" of the CRAY which counts very efficiently the number of ones in a word.

Due to the multi-spin-coding arrangement only chain lengths that are integer multiples of 64 can be considered. In fact, in order to take into account correctly the neighbors at the (periodic) boundaries of the chain our programme allows for lengths $N = 64 \cdot (4 + n)$ with an integer n . If n is also an integer multiple of 64 then the inner loop of the programme vectorizes best and we attain speeds of 500 million updates per second.

3. The First Transient

One way to study transients is to look at the time dependence of the magnetization as was done in Ref. 7. In Fig. 2a we see the decay of $m(t)$ for $p = \frac{1}{2}$ for different sizes N . The curves collapse so well that on the scale of the plot they are totally indistinguishable from each other. This is confirmed in Fig. 2b where we see $\overline{m}(250)$, i.e. the magnetization averaged over the last 50 time-steps against N^{-1} . Within the statistical error bars the value is constant and we obtain

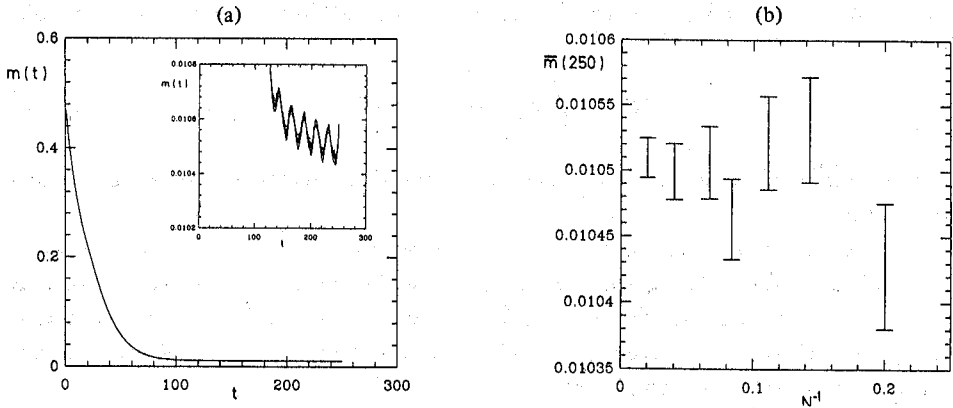


Fig. 2. Magnetization for $p = \frac{1}{2}$ for chain lengths $N = 448, 576, 768, 960, 1600$ and 3200 calculating the statistical average over 10^5 initial configurations and one configuration of $N = 73, 728, 256$. In (a) we see the time dependence. The curves for different sizes fall perfectly on top of each other; only the blow-up of the insert shows some statistical fluctuations on top of an oscillation. In (b) we see the dependence of $\bar{m}(250)$ on the inverse chain length N^{-1} with statistical error bars.

$\bar{m}(250) = 0.01050 \pm 0.00005$. We conclude that for this transient the size dependence is negligible. The insert of Fig. 2a shows in a blow-up that the magnetization actually oscillates.

Another interesting question is the dependence of the transient on the initial configuration. More precisely, we want to see how the results of Fig. 2 change if the fraction of ones in the initial configuration does not have anymore an average value of 0.5. In Fig. 3 we see the magnetization for different concentrations p of ones at $t = 0$. We see from Fig. 3a that only if $p < 0.4$ or $p > 0.8$ the behaviour differs within statistical error bars from the one shown in Fig. 2a. In these cases the decay of $m(t)$ is not smooth anymore. The wiggles one sees for instance for $p = 0.1$ at short times are not statistical noise but are entirely reproduced when the seed of the random number generator is changed or the statistics is increased. We therefore have a complicated structure in the behaviour of $m(t)$ as it was also found to occur in other automata.⁷ For longer times we see in Fig. 3a clearly a periodicity which is more pronounced if p is close to 0 or to 1. This periodicity is probably due to the fact that for large or small p only few types of gliders survive.

In Fig. 3b we see how $\bar{m}(250)$ depends on p . At $p = 0$ and at $p = 1$ the final magnetization vanishes. From the insert one sees that it is not clear if the magnetization does already vanish for finite, small values, i.e. if there is a phase transition at finite p as found in other cases.¹⁰ This would however be astonishing because our system is one-dimensional. It is interesting to note that the magnetization has a maximum at about $p = 0.93$ and that the curve in Fig. 3b is asymmetric with respect to a transformation $p \rightarrow 1 - p$.

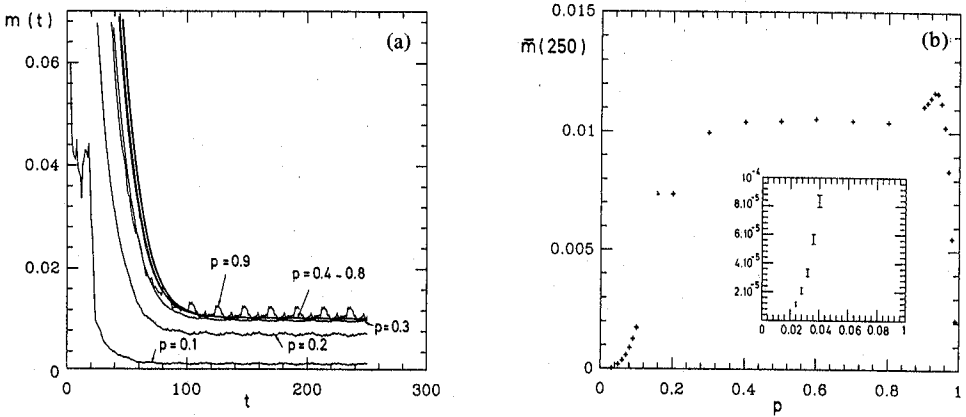


Fig. 3. Magnetization for chain length $N = 6400$ and statistical averages over 10^4 initial configurations for different concentrations p of ones in the initial configuration. In (a) we see in a plot the time evolution of the magnetization; the curves for $p = 0.4, 0.5, 0.6, 0.7$ and 0.8 fall on top of each other. (b) shows the dependence on p of the magnetization $\bar{m}(250)$ averaged between $t = 200$ and $t = 250$. The insert in (b) shows a blow-up of the behaviour for small values of p .

Another way to study the dependence on the initial configuration is to compare the time evolution of a configuration with that of a very slightly modified one. For this purpose one creates a copy $\{\sigma'\}$ of the initial configuration $\{\sigma\}$ and damages the copy in just one site i , i.e. one sets $\sigma'_i = -\sigma_i$. Next, one monitors the time evolution of the total number of damaged sites $\Delta_{\text{tot}}(t)$ averaged over many initial configurations. This is shown in Fig. 4 for various values of p . We see that first the damage spreads and then it decays for $p = \frac{1}{2}$ to a value $\Delta_{\text{tot}}(250) \approx 1.39$. Similarly as for the magnetization of Fig. 3a we see that for large and for small p the curves are not smooth but have quite some structure which is clearly distinguishable from the statistical noise.

The distance ξ that the damage can spread away from the site it was at $t = 0$ is the correlation length that determines the first transient because it is the distance over which a perturbation in the initial configuration can at best propagate. We measured ξ as the distance of the farthest site that was at least damaged once during the first 200 time-steps from the site which was damaged at time $t = 0$ averaged over many initial configurations. In fact, we measured separately the distance of the farthest site to the left and the distance of the farthest site to the right. In Fig. 5 we see these distances as a function of p . For all points the statistical error bars are smaller than the size of the symbols used in the figure. For small and large p this distance goes to zero. At $p = \frac{1}{2}$, i.e. the usual random initialisation, we find $\xi = 26.9 \pm 0.1$. Physically this means that if one places two perturbations at $t = 0$ at a distance larger than 2ξ they will not feel each other's effect. That means that essentially regions of length $N_0 = 2\xi$ are decoupled from each other, which explains

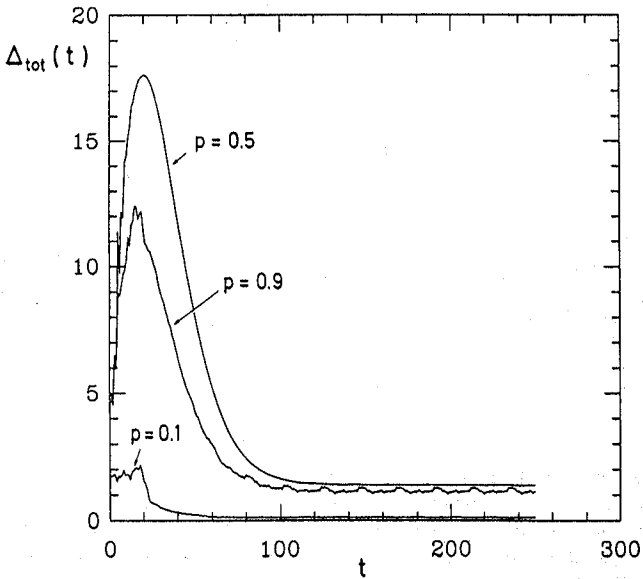


Fig. 4. Total number of damaged sites $\Delta_{tot}(t)$ as a function of time for $N = 320$ averaging over 10^6 configurations for different values of p .

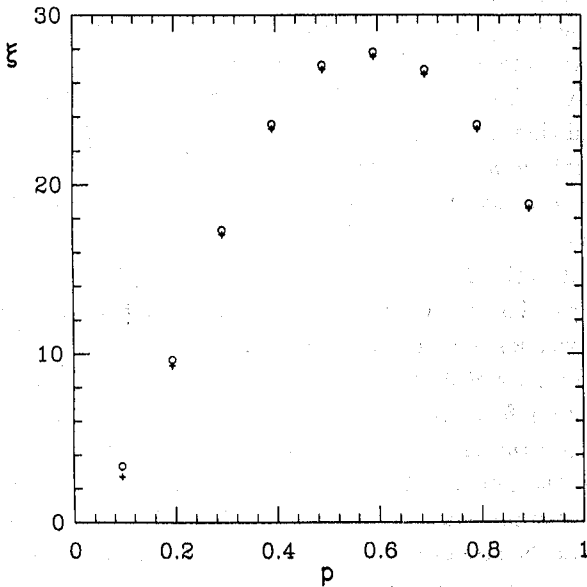


Fig. 5. Average distance ξ of damage spreading as a function of p for $N = 320$ averaged over 10^5 configurations. Crosses are the distances to the left, open circles are the distances to the right of the initial location of the damage.

the lack of size effects seen in Fig. 2. If one supposes that information advances by one site per time-step, this also explains why the first transient abruptly terminates at about 100 time-steps.

Since a small initial perturbation only propagates over a finite distance our system is not chaotic. It is, however, still interesting to see how the damage $\Delta(t)$ at time t as defined in Eq. (2) depends on the amount of initial damage $\Delta(0)$, i.e. not to consider only infinitesimally small damage as we considered up to now. For this purpose we now randomly damage a fraction $\Delta(0)$ of sites of the initial configuration, i.e. the second configuration $\{\sigma'\}$ is made by copying the first configuration $\{\sigma\}$ and changing the variable of each site with probability $\Delta(0)$. In Fig. 6 we see the time evolution of the damage for different values of $\Delta(0)$. We see that the curves collapse on top of each other except for very short times. For all initial damages we obtain $\Delta(250) = 0.024 \pm 0.002$.

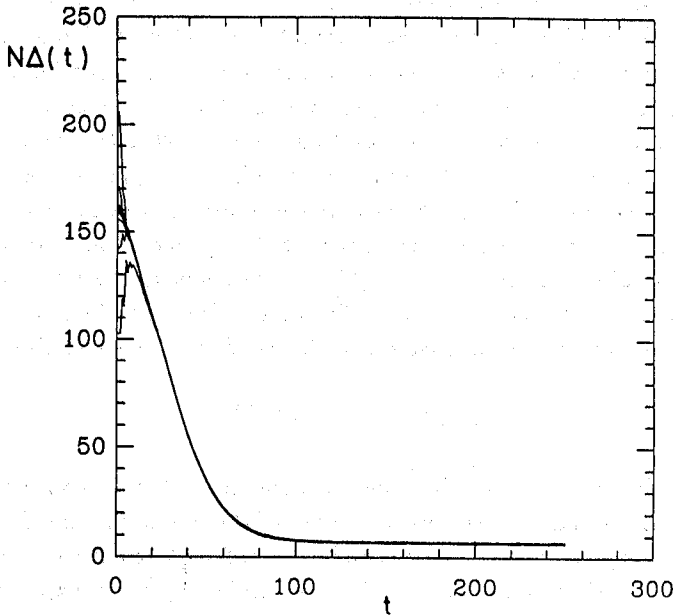


Fig. 6. Number of damaged sites $N\Delta(t)$ as function of time for initial damages $\Delta(0) = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8,$ and 0.9 obtained for $N = 320$ and $p = \frac{1}{2}$ averaging over 1000 initial configurations. Only at short times the curves for different $\Delta(0)$ can be distinguished.

4. The Glider Phase

After the first transient one finds only gliders of various types as can be seen in Fig. 7. The different types of gliders and the probabilities with which they occur have been studied.³ Some gliders move with a certain velocity. As seen in Fig. 8

the moving gliders eventually collide with the neighboring glider and usually both gliders disappear after the collision. Figure 7 suggests that only immobile gliders survive at the end.

In Fig. 9 we see how the magnetization evolves in the glider phase over a long time (200,000 time-steps). The curves from different chain lengths again fall on top of each other within their statistical error bars. We see that the magnetization decays also in the glider phase forming a second transient but over a much larger time-scale than for the first transient. After a time of about 30,000 time-steps the magnetization saturates at a value of $m^* = 0.00690 \pm 0.00005$.

This second transient comes from the collisions and subsequent annihilations of gliders. When all moving gliders have disappeared through collisions only immobile gliders are left over and one expects a periodic magnetization. The asymptotic long-time behaviour is therefore a periodic configuration with an average fraction m^* of ones. The period is finite if the number of possible different immobile gliders is finite, which seems to be the case in our automaton. Therefore the long-time behaviour is indistinguishable from that characterizing class 2 automata.

The second transient can be understood if one uses the fact found in the previous section that for distances more than N_0 sites apart the appearances of gliders are uncorrelated. If one calls q the probability that a unit of N_0 sites produces a glider then $(1-q)^k q$ is the probability of finding two neighboring gliders at k units apart. Let us call v the velocity of a glider measured in units of sites per time-step. The maximum number of units of N_0 sites apart that a glider of velocity v can hit a neighbor in a time t is therefore $k^* = tv/N_0$. Consequently the fraction of gliders annihilated through collision with gliders of velocity v is proportional to $q \sum_i^{k^*} (1-q)^i$ and therefore the decrease of magnetization $\Delta m(t)$ due to these collisions obeys

$$\Delta m(t) = g(1-q)^{k^*+1} \quad (4)$$

where g is the fraction of ones that have become zero due to collisions after all collisions have taken place. Therefore the decay of the magnetization due to gliders of one velocity is exponential. If one has gliders of different velocities v_1, v_2, \dots then the decay is a sum of exponentials: $\Delta m(t) = g_1 \exp(-\alpha v_1 t) + g_2 \exp(-\alpha v_2 t) + \dots$ where $G = g_1 + g_2 + \dots$ is the total decay of magnetization within the second transient and $\alpha = \ln(1-q)/N_0$. In our case $G \approx 0.0036$, $N_0 \approx 55$ and $q \approx 0.15$ for $p = \frac{1}{2}$.

In Fig. 9b we show a semi-logarithmic plot of the decay of the magnetization within the second transient as function of time. Indeed the data could well be a superposition of various straight lines as one would expect from the above argument.

5. Conclusion

Although a clear definition of class 4 automata is not known to us the examples for this class have been characterized by rare structures (gliders in one dimension,

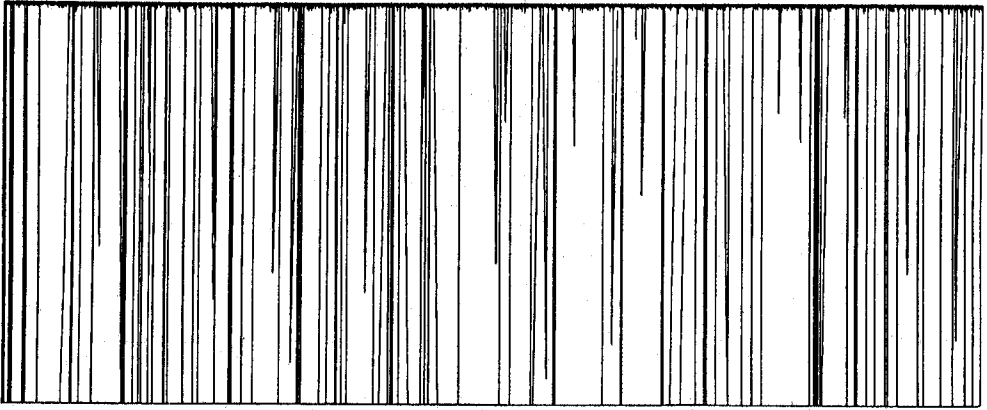


Fig. 7. One configuration of $N = 96000$ sites evolving over 5000 time-steps ($p = \frac{1}{2}$).

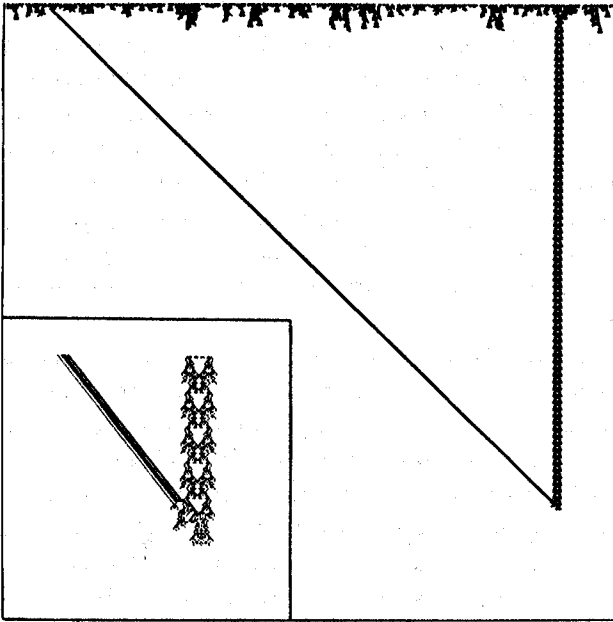


Fig. 8. System of $N = 1920$ sites evolving over 2000 time-steps. The insert is a blow-up of the collision and annihilation region.

in two dimensions there exist also glider guns) which can move and collide and seem to perform an everlasting intricate game of information exchange. For this reason they have been evocated as Turing machines.

The analysis we made of the one-dimensional totalistic rule 20 automaton revealed the following scenario: After a first transient characterized by a finite correlation length ($\xi = 26.9$ for $p = \frac{1}{2}$) and a finite time only gliders are left. The density

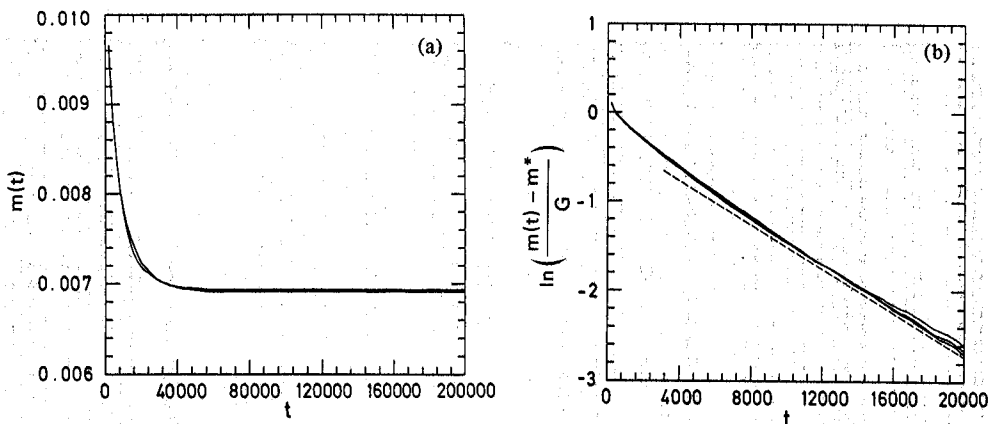


Fig. 9. (a) Magnetization $m(t)$ and (b) $\ln((m(t) - m^*)/G)$ as function of time for $p = \frac{1}{2}$. One sees three curves on top of each other: two independent configurations of $N = 73, 728, 256$ sites and the average over 8000 configurations of $N = 6400$ sites. The dashed straight line in (b) is a guide to the eye.

of gliders is so low (residual magnetization ≈ 0.0105 for $p = \frac{1}{2}$) that compared to the small length scale of the first transient one can essentially consider their type and starting position to be chosen randomly and independently from the others. Most gliders are fixed in space. The small fraction of moving gliders collides typically after several thousand time-steps with a neighbor and in most case both gliders annihilate. After about 30,000 time-steps only immobile gliders remain; the magnetization has sunk to 0.0069 for $p = \frac{1}{2}$. If the number of possible different gliders is finite one has then a stable, periodic configuration as the ones that characterize class 2 automata.

We believe that the above scenario might be generic for automata called class 4. In two dimensions one has as example for a class 4 rule the Game of Life and there too, moving and immobile structures appear with a given density. We think that, the moving structures will eventually collide with other ones and if anything is left at the end it will be a set of spatially constrained, periodic configurations. Also the three-dimensional Games of Life¹¹ should have similar behaviour. In this way, it may be after all, that class 4 automata are just class 2 automata (or class 1 automata if the final magnetization vanishes) with two different transients.

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