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ON ANALYTIC AND PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR OSCILLATORS

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We use the equations of motion for a particle moving in one dimension under the action of forces which vary cubically with displacement to discuss differential equations associated with double-periodic Jacobian elliptic functions. We show that analytic solutions for this dynamical system can be given in terms of Jacobian elliptic functions. These periodic functions are natural extensions of the well-known trigonometric functions.

1. Introduction

In this Letter we present analytic solutions of the differential equation

$$\ddot{x} + \alpha + \beta x + \gamma x^2 + \varepsilon x^3 = 0\,, \tag{1}$$

where α, β, γ and ε are constant parameters. The solutions are given in terms of Jacobian elliptic functions. The problem as defined by Eq. 1 originates from a lecture given in 1860 by Weierstrass and was recently reconsidered in a very interesting paper by Reynolds [1989]. The existence of analytical solutions for nonlinear differential equations such as Eq. 1 is of interest today because, when properly driven or modified to include dissipation, they are used to model a number of physical situations which present limit-cycle behavior and for which chaotic dynamics is known to occur; for example, see the book by Thompson & Stewart [1986] or any recent book on nonlinear dynamics. In particular, Eq. 1 above contains the cubic nonlinearity characteristic of the Duffing oscillator. As mentioned by Reynolds [1989], even though analytic solutions for Eq. 1 must have been known for a long time, their derivations are not trivial and not easily found in the literature. Moreover, the final forms of the solutions can be quite complicated in some cases [Reynolds, 1989]. Reynolds presented a derivation of a solution of the above equation in terms of the Weierstrassian elliptic function [Byrd & Friedman, 1971] $\mathcal{P}(t) \equiv \mathcal{P}(t, g_1, g_2)$. We believe solutions in terms of Jacobian elliptic functions are easier to obtain and "more transparent" than the equivalent ones in terms of Weierstrassian functions. Solutions of Eq. 1 also appear in the semiclassical Jaynes–Cummings model without the standard rotating-wave approximation, describing the interaction of N two-level atoms with a single mode electromagnetic field inside a resonant cavity [Kujawski, 1987; Kujawski & Munz, 1989; Jelenska-Kuklinska & Kus, 1990], as well as in the description of some complicated codimension-2 phenomena present in some laser plasmas [Braun *et al.*, 1992; Gallas, 1993].

As is known, elliptic functions defined by

Weierstrass and Jacobi are related to each other by the formula

$$\mathrm{sn}(\gamma t) = \frac{\gamma}{\sqrt{\mathcal{P}(t) - e_3}}\,,$$
$$\mathrm{cn}(\gamma t) = \sqrt{\frac{\mathcal{P}(t) - e_1}{\mathcal{P}(t) - e_3}}\,, \qquad (2)$$
$$\mathrm{dn}(\gamma t) = \sqrt{\frac{\mathcal{P}(t) - e_2}{\mathcal{P}(t) - e_3}}\,,$$

where in these three equations $\gamma^2 = e_1 - e_3$ (not the arbitrary γ of Eq. 1) and all further symbols are defined by Byrd & Friedman [1971]. However, unlike the Jacobian function sn t, which has a simple pole, Weierstrass' elliptic function $\mathcal{P}(t)$ has a pole of order 2 in any primitive period-parallelogram. Our motivation in looking for solutions of Eq. 1 in terms of Jacobian rather than Weierstrass elliptic functions, is that the former can be regarded as natural "extensions" of the familiar trigonometric functions. By properly investigating the behavior of the periodic solutions of Eq. 1, one may hope to gain insight into the general behavior of the solutions of other commonly used nonlinear differential equations. Recall that analytical solutions for nonlinear differential equations are rare.

Altogether there are twelve Jacobian elliptic functions which can be conveniently defined on a canonical lattice. They depend on a parameter m varying continuously between $m = 0$ (trigonometric limit) and $m = 1$ (hyperbolic limit). They are doubly periodic meromorphic functions with a real period given by $4K(m)$, $K(m)$ being the complete elliptic integral of the first kind. For more on the properties of Jacobian elliptic functions, see for example Byrd & Friedman [1971], Abramowitz & Stegun [1972] or Neville [1951]. To simplify the notation, we will sometimes abbreviate sn $t \equiv \mathrm{sn}(t, m)$, etc.

We consider the differential equations obeyed by Jacobian elliptic functions. Differential equations frequently quoted in standard texts for Jacobian elliptic functions are not the most convenient ones for applications. For example, the standard textbook differential equations are

$$\left\{\frac{d(\mathrm{sn}\,t)}{dt}\right\}^2 = (1 - \mathrm{sn}^2 t)(1 - m\,\mathrm{sn}^2\,t)\,, \qquad (3a)$$

$$\left\{\frac{d(\mathrm{cn}\,t)}{dt}\right\}^2 = (1 - \mathrm{cn}^2 t)(1 + m\,\mathrm{cn}^2\,t)\,, \qquad (3b)$$

$$\left\{\frac{d(\mathrm{dn}\,t)}{dt}\right\}^2 = (1 - \mathrm{dn}^2 t)(\mathrm{dn}^2\,t - 1 + m)\,. \qquad (3c)$$

The appealing point is that they are all of first order. However, they involve multivalued square-root functions. For $m = 0$ it is not even possible to correctly recover from them the trivial set of formulas

$$\frac{d(\sin t)}{dt} = \cos t\,, \quad \frac{d(\cos t)}{dt} = -\sin t\,. \qquad (4)$$

This example shows that, rather than properly being "differential equations," Eqs. 3(a–c) refer to something closer to "differentiation rules." For application in physical problems, these equations are rather limited if not at all useless. Other differential equations are therefore required. Table 1 shows the first and second derivatives of the twelve Jacobian elliptic functions. It is interesting to point out that even modern surveys of elliptic functions such as the recent book by Lawden [1989] still lists the "standard" differential equations mentioned above. The present author holds the opinion that the differential equations as defined in Table 1 are more useful for applications. Below we show how to use Table 1 to solve Eq. 1.

The first derivatives can be found in almost any book on elliptic functions although not always in the most simplified form (see Byrd & Friedman [1971], for example). First derivatives are always proportional to the product of the corresponding copolar trio. From the second derivatives one sees that Jacobian elliptic functions obey very simple second order nonlinear differential equations. These equations may also be obtained by working with radicals. However, we believe the direct calculation of second derivatives to be a convenient way of obtaining the essence of what is needed to solve Eq. 1. Second derivatives also stress the fact the Jacobian functions are periodic solutions of a harmonic oscillator perturbed by a cubic term. We have not seen expressions for such derivatives in the literature before. From Table 1, one clearly sees that rather than first-order, it is best to work with second-order differential equations for Jacobian functions. Note that the coefficients of the equations for sn t and cd t as well as those of dc t and ns t are the same.

By changing variable in Eq. 1, according to $y = px + q$, one obtains (with p, q constants)

$$\ddot{y} + p^4\varepsilon y^3 + (3q\varepsilon + \gamma)p^3 y^2 + (3q^2\varepsilon + 2q\gamma + \beta)p^2 y + p(q^3\varepsilon + q^2\gamma + q\beta + \alpha) = 0\,, \qquad (5)$$

Table 1. The first two derivatives of the twelve Jacobian elliptic functions.

Function	First Derivative	Second Derivative
$\mathrm{sn}(t, m)$	$\mathrm{cn}(t, m)\mathrm{dn}(t, m)$	$2m\,\mathrm{sn}^3(t, m) - (1+m)\mathrm{sn}(t, m)$
$\mathrm{cn}(t, m)$	$-\mathrm{sn}(t, m)\mathrm{dn}(t, m)$	$-2m\,\mathrm{cn}^3(t, m) - (1-2m)\mathrm{cn}(t, m)$
$\mathrm{dn}(t, m)$	$-m\,\mathrm{sn}(t, m)\mathrm{cn}(t, m)$	$-2\,\mathrm{dn}^3(t, m) + (2-m)\mathrm{dn}(t, m)$
$\mathrm{sd}(t, m)$	$\mathrm{cd}(t, m)\mathrm{nd}(t, m)$	$-2m(1-m)\mathrm{sd}^3(t, m) - (1-2m)\mathrm{sd}(t, m)$
$\mathrm{cd}(t, m)$	$-(1-m)\mathrm{sd}(t, m)\mathrm{nd}(t, m)$	$2m\,\mathrm{cd}^3(t, m) - (1+m)\mathrm{cd}(t, m)$
$\mathrm{nd}(t, m)$	$m\,\mathrm{sd}(t, m)\mathrm{cd}(t, m)$	$-2(1-m)\mathrm{nd}^3(t, m) + (2-m)\mathrm{nd}(t, m)$
$\mathrm{sc}(t, m)$	$\mathrm{nc}(t, m)\mathrm{dc}(t, m)$	$2(1-m)\mathrm{sc}^3(t, m) + (2-m)\mathrm{sc}(t, m)$
$\mathrm{nc}(t, m)$	$\mathrm{sc}(t, m)\mathrm{dc}(t, m)$	$2(1-m)\mathrm{nc}^3(t, m) - (1-2m)\mathrm{nc}(t, m)$
$\mathrm{dc}(t, m)$	$(1-m)\mathrm{sc}(t, m)\mathrm{nc}(t, m)$	$2\,\mathrm{dc}^3(t, m) - (1+m)\mathrm{dc}(t, m)$
$\mathrm{ns}(t, m)$	$-\mathrm{ds}(t, m)\mathrm{cs}(t, m)$	$2\,\mathrm{ns}^3(t, m) - (1+m)\mathrm{ns}(t, m)$
$\mathrm{cs}(t, m)$	$-\mathrm{ns}(t, m)\mathrm{ds}(t, m)$	$2\,\mathrm{cs}^3(t, m) + (2-m)\mathrm{cs}(t, m)$
$\mathrm{ds}(t, m)$	$-\mathrm{cs}(t, m)\mathrm{ns}(t, m)$	$2\,\mathrm{ds}^3(t, m) - (1-2m)\mathrm{ds}(t, m)$

which for $q = -\gamma/(3\varepsilon)$ reduces to

$$\ddot{y} + p^4\varepsilon y^3 + p^2(\beta - \gamma^2/(3\varepsilon))y + p[\alpha - \gamma\beta/(3\varepsilon) + 2\gamma^3/(27\varepsilon^2)] = 0\,. \qquad (6)$$

These expressions show that, by properly chosing q, one might get rid of the quadratic or of the constant term but not of both simultaneously for arbitrary values of α, β, γ and ε, although this might happen for particular choices of them. Therefore "bare" Jacobian functions cannot be expected to be solutions of Eq. 1 for arbitrary values of the parameters. Solutions however can now be easily found by adding a constant term to any of the twelve functions in Table 1. This procedure is in sharp contrast with the solution of linear differential equations where the simple addition of constant terms only affects solutions in a trivial way.

Writing one of the solutions of Eq. 1 in the form (with a, b constants)

$$x(t) = a + b\,\mathrm{sn}[\omega(t - t_0), m]\,, \qquad (7)$$

where t_0 is an arbitrary initial value, it is easy to obtain from Table 1 that Eq. 8 is valid.

By equating the coefficients of x^i between Eqs. 1 and 8 one obtains a system of nonlinear equations that allows one to express $(\alpha, \beta, \gamma, \varepsilon)$ as functions of (a, b, ω, m) and vice-versa. Although there is a large family of possible solutions, since m is restricted to the interval $[0, 1]$ we do not expect to find solutions for every arbitrary set $(\alpha, \beta, \gamma, \varepsilon)$. The precise delimitation of all allowed parameter intervals is not a trivial task and will not be attempted here. Table 2 summarizes the twelve nonlinear systems of equations that are obtained by considering every one of the functions in Table 1 in the *Ansatz* of Eq. 7. From this table one sees that solutions always require $\gamma = -3a\varepsilon$.

In conclusion, we have shown that the nonlinear differential equation (1) has periodic and analytical solutions that can be conveniently written in terms of Jacobian elliptic functions. We showed that explicit solutions for a set of parameters $(\alpha, \beta, \gamma, \varepsilon)$ involve the solution of some nonlinear systems of equations. These systems of equations are summarized in Table 2. Jacobian elliptic functions are natural but nontrivial extensions of the familiar trigonometric and hyperbolic functions. In a broad sense, the oscillator defined by Eq. 1 is more fundamental then the familiar harmonic oscillator and contains it as a particular case. Some applications of elliptic functions as models of dynamical systems

$$\ddot{x} + \omega^2(1+m)(x-a) - 2m\frac{\omega^2}{b^2}(x-a)^2 = \ddot{x} - \frac{\omega^2}{b^2}a[b^2(1+m) - 2ma^2] + \frac{\omega^2}{b^2}[b^2(1+m) - 6ma^2]x + \frac{6m\omega^2 a}{b^2}x^2 - \frac{2m\omega^2}{b^2}x^3 = 0\,. \qquad (8)$$

Table 2. Nonlinear systems of equations defining the parameters of the solutions of Eq. 1.

Function	$\alpha b^2/\omega^2$	$\beta b^2/\omega^2$	$\gamma b^2/\omega^2$	$\varepsilon b^2/\omega^2$
sn	$-a[b^2(1+m)-2ma^2]$	$b^2(1+m)-6ma^2$	$6ma$	$-2m$
cn	$-a[b^2(1-2m)+2ma^2]$	$b^2(1-2m)+6ma^2$	$-6ma$	$2m$
dn	$a[b^2(2-m)-2a^2]$	$-b^2(2-m)+6a^2$	$-6a$	2
sd	$-a[b^2(1-2m)+2m(1-m)a^2]$	$b^2(1-2m)+6m(1-m)a^2$	$-6m(1-m)a$	$2m(1-m)$
cd	$-a[b^2(1+m)-2ma^2]$	$b^2(1+m)-6ma^2$	$6ma$	$-2m$
nd	$a[b^2(2-m)-2(1-m)a^2]$	$-b^2(2-m)+6(1-m)a^2$	$-6(1-m)a$	$2(1-m)$
sc	$a[b^2(2-m)+2(1-m)a^2]$	$-b^2(2-m)-6(1-m)a^2$	$6(1-m)a$	$-2(1-m)$
nc	$-a[b^2(1-2m)-2(1-m)a^2]$	$b^2(1-2m)-6(1-m)a^2$	$6(1-m)a$	$-2(1-m)$
dc	$-a[b^2(1+m)-2a^2]$	$b^2(1+m)-6a^2$	$6a$	-2
ns	$-a[b^2(1+m)-2a^2]$	$b^2(1+m)-6a^2$	$6a$	-2
cs	$a[b^2(2-m)+2a^2]$	$-b^2(2-m)-6a^2$	$6a$	-2
ds	$-a[b^2(1-2m)-2a^2]$	$b^2(1-2m)-6a^2$	$6a$	-2

will be published elsewhere. One example is in Gallas [1992].

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References

Abramowitz, M. & Stegun, I. [1972] *Handbook of Mathematical Functions* (Dover, New York).

Bowman, F. [1961] *Introduction to Elliptic Functions with Applications*, reprint of the original 1953 edition (Dover New York).

Braun, T., Lisboa, J. & Gallas, J. A. C. [1992] "Evidence of Homoclinic Chaos in the Plasma of a Glow Discharge," *Phys. Rev. Lett.* **68**, 2770–2773.

Byrd, P. F. & Friedman, M. D. [1971] *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer-Verlag, Berlin) pp. 308–315.

Gallas, J. A. C. [1992] "Feigenbaum's constant for meromorphic functions," *Int. J. Mod. Phys.* **C3**, 553–560.

Gallas, J. A. C. [1993], in preparation.

Jelenska-Kuklinska, M. & Kus, M. [1990] "Exact solution in the semiclassical Jaynes-Cummings model without the rotating-wave approximation," *Phys. Rev.* **A41**, 2889–2891.

Kujawski, A. [1987] "Exact periodic solution in the Jaynes-Cumming model without the rotating-wave approximation," *Phys. Rev.* **A37**, 1386–1387.

Kujawski, A. & Munz, M. [1989] "Exact periodic solutions and chaos in the semiclassical Jaynes-Cummings model," *Z. Phys.* **B76**, 273–276.

Lawden, D. [1989] *Elliptic Functions and Applications* (Springer-Verlag, New York).

Neville, E. H. [1951] *Jacobian Elliptic Functions* (Claredon Press, Oxford).

Reynolds, M. J. [1989] "An exact solution in non-linear oscillations," *J. Math. Phys.* **A22**, L723–726.

Thompson, J. M. T. & Stewart, H. B. [1986] *Nonlinear Dynamics and Chaos* (Wiley, Chichester).