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Periodicity versus chaos in the dynamics of cobweb models [1]

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Abstract

This paper studies the dynamics of economic models when two parameters are simultaneously varied and concentrates on the dynamics of the cobweb model with adaptive expectations. The simultaneous variation of several parameters intervening in economic processes is a very realistic situation of interest. Using reliable numerical methods, we argue that extended fractal sets in *parameter space* are relatively common characteristics to be expected for models typically used to describe economic processes. In addition, by choosing appropriate paths in the parameter space, it is possible to observe several different routes to chaos.

JEL classification: E32

Keywords: Non linear cobweb model; Bifurcations; Chaos; Iso-period plot; Chaos plot

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1. Introduction

The use of nonlinear models based on economic themes of growth and business fluctuations to explain periodic and nonperiodic behavior of economic time series is by now a well established research activity. Readers of this journal will be familiar with the examples provided in Medio (1987) and Day (1991). Other collections also have appeared. One of the most interesting aspects of this line of research is the ability to trace the effect of parameter changes that have economic meaning on the implied economic behavior, and in particular, how the quantitative properties of the time series data that can be generated by a model can shift from convergence to stationary states or stable cycles to nonperiodic, highly irregular or chaotic behavior.

The primary technique used so far in these studies is the ''bifurcation diagram'' which provides a numerical (and necessarily approximate) picture of how the ''attractor'' of the trajectory changes when a single parameter is varied. In this paper we show how such studies can be extended when two parameters are varied simultaneously, using some reliable numerical methods called the *Iso-Period Plot* and *Chaos Plot* methods. For illustration, we use the cobweb model with adaptive expectations described in this journal by Hommes (1994).

2. The cobweb model with adaptive expectations

Let p_t be the price, $\hat{p}_t$ the expected price, q_t^d the demand for goods and q_t^s the supply for goods, all at time t. The cobweb model with adaptive expectations is given by

$$q_t^d = D(p_t) \text{ (Demand function)} \tag{1}$$

$$q_t^s = S(\hat{p}_t) \text{ (Supply function)} \tag{2}$$

$$q_t^d = q_t^s \text{ (Market clearing equation)} \tag{3}$$

$$\hat{p}_t = \hat{p}_{t-1} + w(p_{t-1} - \hat{p}_{t-1}),\ 0 \leq w \leq 1 \text{ (Adaptive expectations formula)} \tag{4}$$

In the traditional version of the cobweb model, it was shown by Artstein (1983) and Jensen and Urban (1984) that chaotic price behavior can occur if the supply or demand curve is non-monotonic, see also Lichtenberg and Ujihara (1989). In the case of linear supply and demand curves, the introduction of adaptive expectations in the cobweb model has a stabilizing effect on the price dynamics, see Nerlove (1958). However, the equilibrium price may still be unstable. Hommes (1994) investigated the price behavior in the case of nonlinear, monotonic supply and demand curves and showed that adaptive expectations need not be stabilizing and prices can be chaotic. For simplicity assume that the demand curve is linearly decreasing, and is given by

$$D(p_t) = a - bp_t,\ b > 0 \tag{5}$$

Concerning the supply, the following two economic assumptions are made: (a) If prices are low then supply increases slowly, because of start-up costs and fixed production costs; and (b) If prices are high then supply increases slowly, because of supply and capacity constraints. The simplest smooth curve satisfying these two assumptions is an S-shaped function $S(\cdot)$ with a unique inflection point $\bar{p}$. By changing coordinates, we may assume that the inflection point of the supply curve to be the new origin. Note that with respect to this new origin both "prices" and "quantities" can be negative. As an example of such an S-shaped supply function we follow Hommes (1994) and choose

$$S_\lambda(x) = \arctan(\lambda x), \lambda > 0 \tag{6}$$

Eqs. (1)–(6) yield a difference equation $x_{t+1} = F_{a,b,w,\lambda}(x_t)$ describing the expected price dynamics, with $F_{a,b,w,\lambda}$ given by

$$F_{a,b,w,\lambda}(x) = -wS_\lambda(x)/b + (1-w)x + aw/b \tag{7}$$

where $a \in \mathbb{R}$, $b > 0$, $0 \le w \le 1$, and $\lambda > 0$.

It is easily verified that the following holds. If $S'_\lambda(0) \le b(1-w)/w$ then the map $F_{a,b,w,\lambda}$ is increasing, and if $S'_\lambda(0) > b(1-w)/w$ then $F_{a,b,w,\lambda}$ has two critical points c_1 and c_2, $c_1 < 0 < c_2$ (that is, c_1 and c_2 are the two points at which the derivative of $F_{a,b,w,\lambda}$ vanishes). Hence, if $S'_\lambda(0) > b(1-w)/w$ then the map $F_{a,b,w,\lambda}$ is increasing on both the intervals $(-\infty, c_1)$ and (c_2, ∞), and $F_{a,b,w,\lambda}$ is decreasing on the interval (c_1, c_2). The map $F_{a,b,w,\lambda}$ has a unique fixed point, which is the equilibrium price corresponding to the intersection point of the supply and demand curves. An important question is: "What can be said about the global dynamics of the model, when the equilibrium price is unstable?" In this direction, Hommes (1994) obtained results concerning period-2 behavior and Li-Yorke chaos in the dynamics of the map $F_{a,b,w,\lambda}$. ("Li-Yorke chaos" is implied when the map $F_{a,b,w,\lambda}$ has a period-m orbit, where m is not a power of 2.) We now are going to investigate how trajectories change when alternative combinations of two of the parameters are changed.

3. Numerical methods: Chaos Plot and Iso-Period Plot methods

Before presenting our numerical methods for deriving qualitative dynamics for two varying parameters, we must review some technical concepts. Let F denote a differentiable one-dimensional map depending on k parameters, where k is a positive integer. For any x_0, write $DF^n(x_0)$ for the derivative of the n^{th} iterate of F at x_0, where n is any positive integer. The **Lyapunov exponent** $\lambda(x_0)$ of the trajectory of x_0 is defined by $\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\log |DF^n(x_0)|$ where we assume that the limit exists. (Here "log" means natural logarithm.) The trajectory of a point x_0 is called a **chaotic trajectory** if (a) the trajectory of x_0 is bounded and is not asymptotic to either a fixed point or a periodic orbit, and (b) the trajectory of

x_0 has a positive Lyapunov exponent. For the map F, a chaotic trajectory is a Li-Yorke chaotic trajectory and vice versa. For a continuous map, a chaotic trajectory is a Li-Yorke chaotic trajectory but the reverse need not be true. Although in practice we can only probe a finite number of iterates, we will often speak of chaotic trajectories. Also we speak loosely of Lyapunov exponents as if they could be computed with precision. We emphasize that a chaotic trajectory exhibits sensitive dependence on initial conditions, that is, arbitrarily close to the initial state x_0 there are points y_0 such that the trajectories of x_0 and y_0 do not stay close to each other forever.

In the literature there exist several (nonequivalent) definitions of the notion "attractor"; for a discussion of this topic, see Milnor (1985). In this paper, we say that a set A on the real line is an **attractor** if (i) A is compact and $F(A) \subset A$, (ii) there exists an open neighborhood U of A such that for each point x in U the distance between $F^n(x)$ and A converges to zero as n goes to infinity, and (iii) A has a dense orbit (that is, there exists a point x_0 in A such that for every (small) open ball B centered at a point in A, there exists a positive integer k such that the k^{th} iterate x_k is contained in B). An attractor A is called a **chaotic attractor** if A contains a chaotic trajectory. Note that if F has a chaotic attractor, then F has sensitive dependence on initial conditions and there are many choices for a Li-Yorke scrambled set. For example, if A is a chaotic attractor for F, then A has positive Lebesgue measure and is an example of a Li-Yorke scrambled set, but there also many invariant Cantor sets (of measure zero) in A and each of them is a Li-Yorke scrambled set. On the other hand, if F has a stable periodic orbit P_m of period $m \neq 2^n$ for any n, and if almost every initial condition will ultimately approach P_m, then the collection of points whose trajectories will not approach P_m is a Li-Yorke scrambled set.

The numerical methods for plotting bifurcation diagrams and corresponding Lyapunov exponent bifurcation diagrams are standard; see Nusse and Yorke (1994). In these diagrams only one parameter is varied while all others are maintained fixed. For illustration, see Fig. 1. We now describe the numerical methods for creating Chaos Plots and Iso-Period Plots, in which two parameters are varied and the other parameters are maintained fixed. The resulting pictures are plots in the parameter space. These methods can be used to detect regions in parameter space in which several attractors coexist and to study the prevalence of periodic versus chaotic dynamics.

3.1. Chaos Plots

Let μ and α be two parameters of a model under investigation. The pair (μ, α) is called a pair of **chaotic parameters** if the map F has a chaotic attractor for these parameter values μ and α. The purpose is to generate a set in the parameter space such that for each point in this set the trajectory through some fixed initial state x_0 is chaotic (that is, it has a positive Lyapunov exponent). In

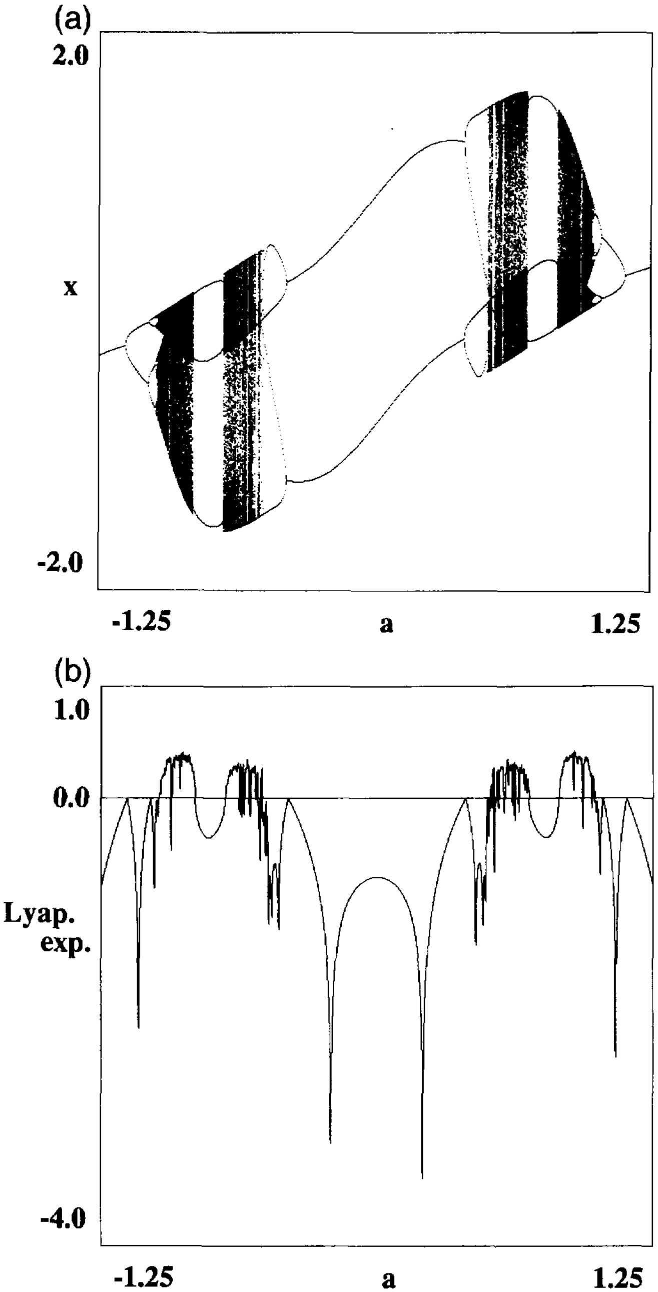


Fig. 1. a – Bifurcation diagram of the cobweb model (8) with b = 0.3, $\lambda = 7$, w = 0.3. b – Lyapunov exponent bifurcation diagram corresponds to Fig. 1a, for the cobweb model (8) with b = 0.3, $\lambda = 7$, w = 0.3. A positive value of the Lyapunov exponent indicates the existence of a chaotic attractor.

other words, we generate the set of parameter values for which the orbit of x_0 converges to a chaotic attractor. A **Chaos Plot** shows pairs of chaotic parameters (μ,α) for the bifurcation parameters μ between μ_L and μ_R and α between α_L and α_R. The objective is to plot (μ,α)-values for which the trajectory through the seed x_0 has a positive Lyapunov exponent. In this paper, Chaos Plots have been constructed as follows. We choose to have a grid of 720 by 720 to be tested. Our criterion is to plot the point (small box) in parameter space if the approximate Lyapunov exponent is at least 0.001 after 10000 iterates of x_0. This method for creating Chaos Plots is implemented in the program *Dynamics* (Nusse and Yorke, 1994).

3.2. Iso-Period Plots

The point (μ,α) in parameter space is called a pair of **period-p parameters** if the map F has a stable period-p orbit for these parameter values. An **Iso-Period-p parameter** set consists of all pairs of period-p parameters (μ,α) for the bifurcation parameters μ between μ_L and μ_R and α between α_L and α_R. If a stable period-p orbit and a stable period-q orbit coexist, then the Iso-Period-p parameter set and the Iso-Period-q parameter set intersect. The purpose is to describe how to generate a set in the parameter space such that for each point in this set the trajectory through some fixed initial state x_0 is approaching a stable period-p orbit (that is, generally it has a negative Lyapunov exponent). In other words, we generate the set of parameter values for which the orbit of x_0 converges to a stable period-p orbit. In this paper, the Iso-Period-p Plots have been constructed by determining periodicities and storing the informations as color Postcript bitmaps as follows. We used a grid of 1200 (horizontal) by 1200 (vertical) to be tested. Our criterion is to plot the point (small box) in parameter space in color p if p is the smallest positive integer for which the trajectory through the seed x_0 has the following property: after the first 1000 iterates of x_0, the distances $|x_{1001+p} - x_{1001}| < \epsilon$, $|x_{1001+2p} - x_{1001+p}| < \epsilon$, and $|x_{1001+3p} - x_{1001+2p}| < \epsilon$, where ϵ is some prescribed positive number. We consider $1 \leq p \leq 64$ and select $\epsilon = 0.0001$. An **Iso-Period Plot** is the union of all Iso-Period-p Plots, where $1 \leq p \leq 64$. From the above it is clear that we compare the successive end points of approximate period-p orbits. The method for creating Iso-Period Plots is the same used by Gallas (1993, 1994) to investigate the structure of the parameter space of the Hénon map.

The two numerical methods above are complimentary. Nusse (1987, 1988)) analyzes a broad class of one-dimensional maps which includes the map $F_{a,b,w,\lambda}$. The map $F_{a,b,w,\lambda}$ has a negative Schwarzian derivative if and only if $w\lambda^2 - b > 0$. Using this fact, we have the following results from these papers, which guarantee that it is sufficient to use just two initial conditions for creating both Chaos Plots and Iso-Period Plots in Section 4.

Theorem 1. Assume $w\lambda^2 - b > 0$. If $F_{a,b,w,\lambda}$ has an attractor, then at least one of the points c_1 or c_2 is contained in the basin of that attractor.

Theorem 2. Assume $w\lambda^2 - b > 0$. If both critical points c_1 and c_2 are contained in the basins of attractors A_1 and A_2, then the trajectory of almost every initial condition x_0 approaches one of these attractors A_1 and A_2.

The first result (Theorem 1) implies that at most two attractors can coexist for the map $F_{a,b,w,\lambda}$. The attractors in the second result (Theorem 2) may coincide, that is, the map $F_{a,b,w,\lambda}$ has one attractor and contains both critical points in its basin. To state the main result for the Chaos Plots and Iso-Period Plots, the results above imply that at most two attractors may coexist for $F_{a,b,w,\lambda}$ and that each attractor will contain at least one of the points c_1 or c_2 in its basin. In particular, if $F_{a,b,w,\lambda}$ has attractor A and contains c_1 in its basin, then the trajectory of c_1 converges to A. The case when c_2 is contained in the basin of attractor A is dealt similarly. Hence, for creating a Chaos Plot, it is sufficient to use initial conditions $x_0 = c_1$ and $x_0 = c_2$ rather than having to explore all initial conditions. Similarly, if A is a period-p orbit, then the trajectory of c_1 (or c_2) asymptotes to a period-p orbit. Hence, for creating an Iso-Period Plot, it is sufficient to use initial conditions $x_0 = c_1$ and $x_0 = c_2$ rather than having to explore all initial conditions. Therefore, we have the following result.

Conclusion. For the cobweb model with adaptive expectations (1)–(6), only two initial conditions are needed for creating Iso-Period Plots and Chaos Plots provided that $w\lambda^2 - b > 0$. These initial conditions are $x_0 = c_1$ and $x_0 = c_2$.

4. Periodic versus chaotic behavior in the dynamics of cobweb models

Under certain conditions, the dynamics of the cobweb model, defined by Eqs. (1)–(6), may exhibit periodic or chaotic behavior. How prevalent are these alternative types of behavior? To give an answer, we investigate parts of the 4-dimensional (a,b,w,λ)-parameter space by using the numerical methods discussed in Section 3. We first consider the four parameter family of maps $F_{a,b,w,\lambda}$: $\mathbb{R} \to \mathbb{R}$ defined by Eqs. (6) and (7). Hence,

$$F_{a,b,w,\lambda}(x) = -w \arctan(\lambda x)/b + (1-w)x + aw/b \tag{8}$$

where $a \in \mathbb{R}$, $b > 0$, $0 \leq w \leq 1$, and $\lambda > 0$.

For Eq. (8) we restrict ourselves to investigate parts of the following slices of the 4-dimensional (a,b,w,λ)-parameter space: (i) $b = 0.3$, $w = 0.3$, (ii) $b = 0.05$, $\lambda = 10$, (iii) $b = 0.25$, $\lambda = 4$, and (iv) $a = 0.8$, $b = 0.25$. We emphasize that for each of these four slices, the condition $w\lambda^2 - b > 0$ appearing in the theorems in Section 3 is satisfied for parameters in the selected regions (specified below). For example, in case (i) we have $b = 0.3$ and $w = 0.3$, and the selected region is

$-1.25 < a < 1.25$ and $2 < \lambda < 52$, so we have $w\lambda^2 - b = 0.3\lambda^2 - 0.3 > 0.9 > 0$ for all (a, λ) in the selected region.

4.1. Slice 1: $b = 0.3$, $w = 0.3$

We first select $\lambda = 7$. We now investigate how the dynamics of the model depends on the height of the demand curve (parameter a), and we assume that a is the bifurcation parameter. Fig. 1a shows a bifurcation diagram with the parameter a as the bifurcation parameter. This figure suggests the following bifurcation scenario. If a is small then there exists a stable equilibrium. If a is increased, then the equilibrium becomes unstable and period doubling bifurcations occur. After infinitely many period doubling bifurcations the price behavior becomes chaotic, as a is increased. Next, after infinitely many period halving bifurcations the price behavior becomes more regular again. A stable period 2 orbit occurs for an interval of a-values, containing $a = 0$. When a is further increased, once more, after infinitely many period doubling bifurcations chaotic behavior arises. Finally, after infinitely many period halving bifurcations, there is a stable equilibrium again, when a is sufficiently large. In particular, there exist parameter values $a_1 < a_2 < a_3 < a_4 = 0 < a_5 < a_6 < a_7$ such that

- If $a < a_1$, then $F_{a,b,w,\lambda}$ has a globally stable fixed point;
- If $a_2 < a < a_3$, then $F_{a,b,w,\lambda}$ has a period 3 orbit;
- If $a = a_4 = 0$, then $F_{a,b,w,\lambda}$ has an unstable fixed point, a stable period-2 orbit, and no other periodic points;
- If $a_5 < a < a_6$, then $F_{a,b,w,\lambda}$ has a period 3 orbit;
- If $a > a_7$, then $F_{a,b,w,\lambda}$ has a globally stable fixed point.

For $a_2 < a < a_3$ or $a_5 < a < a_6$, the map $F_{a,b,w,\lambda}$ is Li-Yorke chaotic, see Li and Yorke (1975). Concerning the bifurcations scenario with respect to the parameter a, the following properties can be shown to hold by combining results from bifurcation theory (see Guckenheimer and Holmes (1983)) and results in Nusse and Yorke (1988). Although the one dimensional maps studied in this paper have one critical point, the results and techniques work for all differentiable maps, since the critical point plays no role in the local bifurcations (cf. Hommes (1994)).

- infinitely many period-doubling bifurcations occur in the parameter intervals (a_1,a_2) and (a_3,a_4).
- infinitely many period-halving bifurcations occur in the parameter intervals (a_2,a_3) and (a_4,a_5).

Recall that increasing the parameter a is just shifting the demand curve vertically upwards. Hence, if the supply curve is shifted vertically upwards, then both infinitely many period doubling and period halving bifurcations occur and periodic and chaotic behavior interchange several times.

Since the map in Eq. (8) is one dimensional, there is just one Lyapunov exponent corresponding to a trajectory. We now investigate how the Lyapunov exponent depends on the parameter a. In Fig. 1b we present a Lyapunov exponent bifurcation diagram with respect to the parameter a. This interval is the same as the one in Fig. 1a. Lyapunov exponent bifurcation diagrams indicate whether the corresponding bifurcation diagram has a stable periodic orbit (negative Lyapunov exponent) or a chaotic attractor (positive Lyapunov exponent).

We consider the region in the parameter space consisting of pairs (a,λ) such that $-1.25 < a < 1.25$, $2 < \lambda < 52$. The objective is to plot the Chaos Plot and Iso-Period Plot for this region as described in Section 3. The resulting Chaos Plot is presented in Fig. 2a. In this case 17% of the grid parameters are chaotic parameters implying that chaos is not so exceptional. The computation for the creation of the Iso-Period Plot is performed exactly as described before in Section 3. The resulting Iso-Period Plot is shown in Fig. 2b. in this figure, the green area consists of pairs of parameter values for which the dynamics will stabilize. The blue area consists of pairs of parameter values for which the dynamics exhibits a period-2 behavior. The light-blue area corresponds to period-3 behavior in the dynamics, the red area corresponds to period-4 behavior in the dynamics, the magenta area corresponds to period-5 behavior, the black area corresponds to period-6 behavior, and the cyan corresponds to period-8 behavior, etc. Our color-coding was chosen to obtain maximal contrast between different areas.

The bifurcation diagram of Fig. 1a and the Lyapunov exponent bifurcation diagram of Fig. 1b correspond to the path $\lambda = 7$ in Figs. 2a and 2b. The figures suggest that periodic behavior and chaotic behavior are complementary, that is, the sum of the area of the periodic parameters and the area of the chaotic parameters equals the area of the region. Unfortunately, there exists no theoretical result which provides a result into this direction. Choosing any path in Fig. 2b, one can predict the period of the periodic attractors that will be encountered when a bifurcation diagram is created by following the parameter(s) along this path. Similarly, one can predict when any attractor which is encountered is chaotic by following the same path in Fig. 2a. Obviously, the structure of the sets of periodic parameters and chaotic parameters is complicated. As is easily seen from Figs. 2a and 2b, there are different routes to chaos in the cobweb model in the current slice of the parameters space.

4.2. Slice 2: $b = 0.05$, $\lambda = 10$

We first select $w = 0.2$, and create a bifurcation diagram by following the attractor. In the resulting bifurcation diagram of Fig. 3a, several phenomena can be observed: existence of a simple attractor with low period, existence of a chaotic attractor (to be checked by the computation of Lyapunov exponents), the existence of a period-3 attractor implying Li-Yorke chaos, and various bifurcations, such as period-doubling and period-halving bifurcation. The corresponding Lyapunov exponent bifurcation diagram is shown in Fig. 3b, and it indicates whether the bifurcation diagram of Fig. 3a has a stable periodic orbit (negative Lyapunov exponent) or a chaotic attractor (positive Lyapunov exponent).

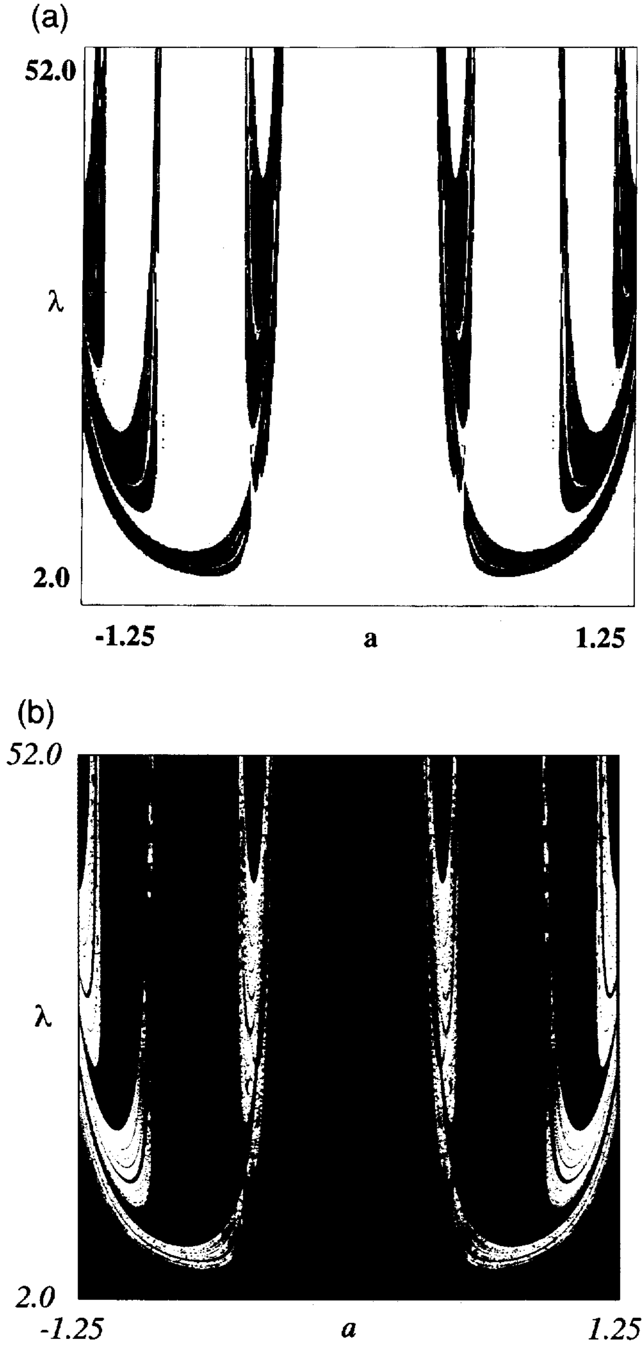


Fig. 2. a – Chaos Plot for cobweb model (8) with b = 0.3, w = 0.3. The black area is the set of parameters for which the model has a chaotic attractor. In this part of the parameter space, chaos is not so exceptional (approximately 17% of the parameters are chaotic parameters). b – Iso-Period Plot for cobweb model (8) with b = 0.3, w = 0.3. The green area indicates that the dynamics will stabilize (period-1 behavior). The dark blue and red area indicate that the dynamics will exhibit a period-2 and period-4 behavior, respectively. Light blue area indicate period-3 behavior. For more details on the colors, see the text.

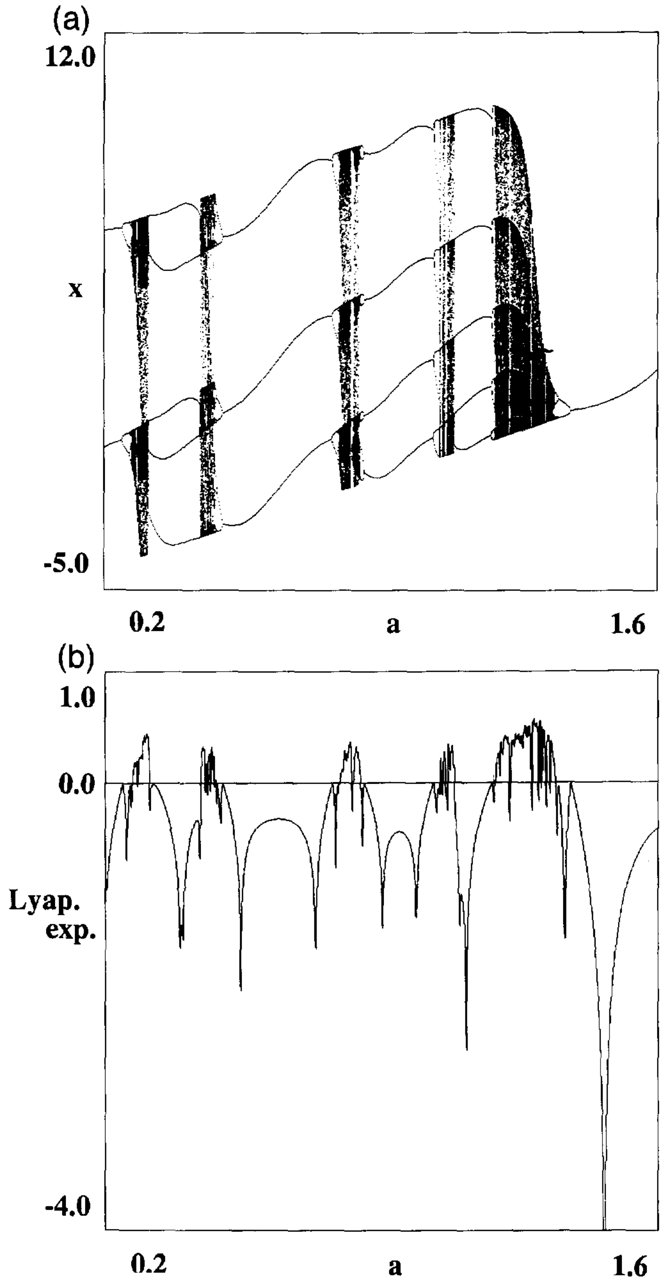


Fig. 3. a – Bifurcation diagram of cobweb model (8) with $b = 0.05$, $\lambda = 10$, $w = 0.2$. b – Lyapunov exponent bifurcation diagram corresponding to Fig. 3a, for the cobweb model (8) with $b = 0.05$, $\lambda = 10$, $w = 0.2$.

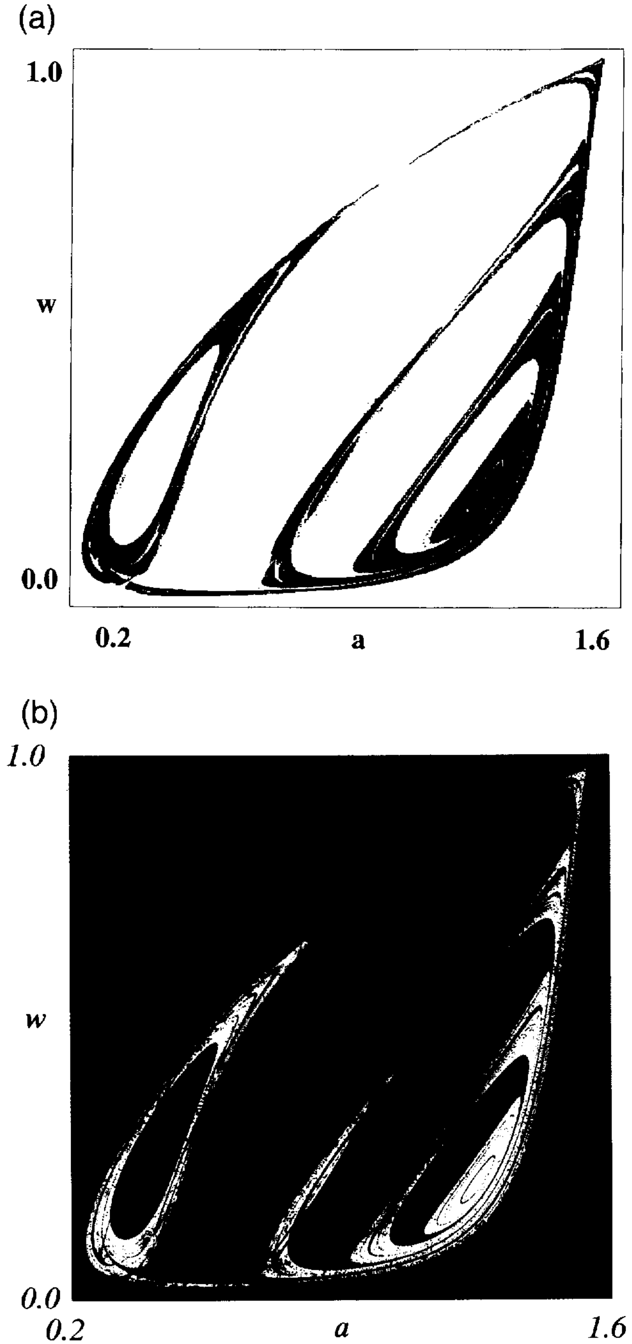


Fig. 4. a – Chaos Plot for cobweb model (8) with $b = 0.05$, $\lambda = 10$. The black area is the set of parameters for which model (8) has a chaotic attractor. In this part of the parameter space, approximately 12% of the parameters are chaotic parameters. b – Iso-Period Plot for cobweb model (8) with $b = 0.05$, $\lambda = 10$. The color coding is the same as in Fig. 2b.

Fig. 4a shows the Chaos Plot for $0.2 < a < 1.6$ and $0 < w < 1$, calculated as described above. In this case approximately 12% of the grid parameters are chaotic parameters. The corresponding Iso-Period Plot is shown in Fig. 4b.

The bifurcation diagram of Fig. 3a and the Lyapunov exponent bifurcation diagram of Fig. 3b correspond to the path $w = 0.2$ in Figs. 4a and 4b. Fig. 4b shows that periodic behavior in the dynamics of the cobweb model is prevalent. In particular, period-1, period-2, period-3, period-4, and period-5 will dominate. Similar as in the first slide, the figures suggest that periodic behavior and chaotic behavior are complementary. Choosing any path in Fig. 4b, one can predict the period of the periodic attractors that will encountered when a bifurcation diagram is created by following the parameter(s) along this path. Similarly, one can predict when any attractor which is encountered is chaotic by following the same path in Fig. 2a. Obviously, the structure of the sets of periodic parameters and chaotic parameters is complicated. As is easily seen from Figs. 4a and 4b, there are different routes to chaos in the cobweb model in the current slice of the parameter space. On the other hand, there also exists reasonable paths for which no chaotic behavior will be observed. In particular, if a path is followed in a unicolor region, then the period of the periodic attractor does not change.

4.3. Slice 3: $b = 0.25$, $\lambda = 4$

Fig. 5a displays the Chaos Plot for $-1.5 < a < 1.5$ and $0.1 < w < 0.9$. In this case approximately 12.5% of the grid parameters are chaotic parameters. The corresponding Iso-Period Plot is shown in Fig. 5b. As one sees, they present consistent characterizations.

Once again, the figures suggest that periodic behavior and chaotic behavior are complementary (that is, the sum of the area of the periodic parameters and the area of the chaotic parameters equals the area of the region). Fig. 5b shows that periodic behavior, in particular period-1, period-2, and period-4 behavior, is prevalent. There are two annuli of parameters (a,w) in this slide for which there is period-4 behavior. Inside these two annuli, the structure of the sets of periodic parameters and chaotic parameters is complicated. For parameters inside these two annuli, the figure suggests that the sets of parameters for which there is periodic behavior in the dynamics for a certain specified period, are annuli. Once again, it is easily seen from Figs. 5a and 5b, that there are different routes to chaos in the cobweb model in the current slice of the parameter space.

4.4. Slice 4: $a = 0.8$, $b = 0.25$

Fig. 6a exhibits the Chaos Plot for $2 < \lambda < 52$ and $0 < w < 1$. In this case approximately 10% of the grid parameters are chaotic parameters. The corresponding Iso-Period Plot is shown in Fig. 6b. Once again, the figures suggest that periodic behavior and chaotic behavior are complementary.

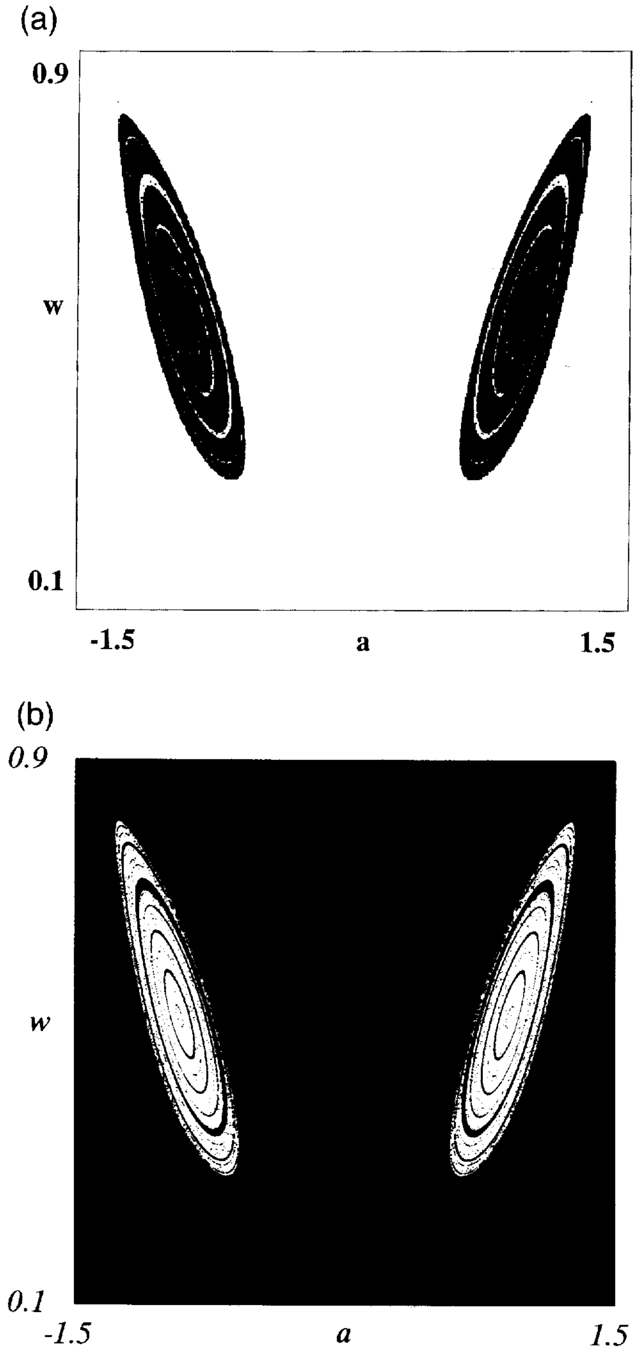


Fig. 5. a – Chaos Plot for cobweb model (8) with $b = 0.25$, $\lambda = 4$. The black area is the set of parameters for which the model has a chaotic attractor. In this part of the parameter space, approximately 12.5% of the parameters are chaotic parameters. b – Iso-Period Plot for cobweb model (8) with $b = 0.25$, $\lambda = 4$. The color coding is the same as in Fig. 2b.

Fig. 6b shows that periodic behavior, in particularly period-1, period-2, period-3, and period-4 behavior, is prevalent. The structure of the sets of periodic parameters and chaotic parameters is quite complicated, but very intriguing. A blow-up of Fig. 6b is given in Fig. 6c. Connected subsets of parameters (λ,w) in this slide for which there is period-m behavior have a quite complicated shape. Gallas (1993, 1994)) coined these components of parameters *shrimps*. (A component is a maximal connected set.) Similarly as in the other cases. It is easily seen from Figs. 6a and 6b, that there are different routes to chaos in the cobweb model in the current slice of the parameter space.

4.5. Summary

In all four slides of the parameter space, one realizes from the Chaos Plots and Iso-Period Plots that the dynamics for any given quadruple of parameters

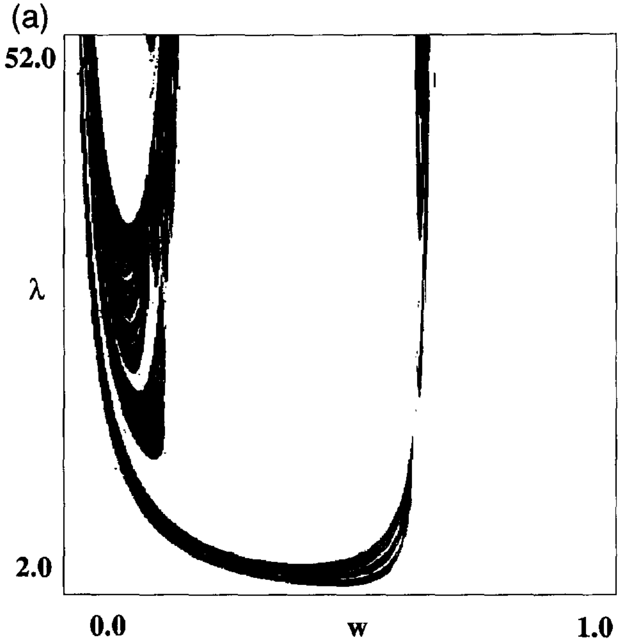
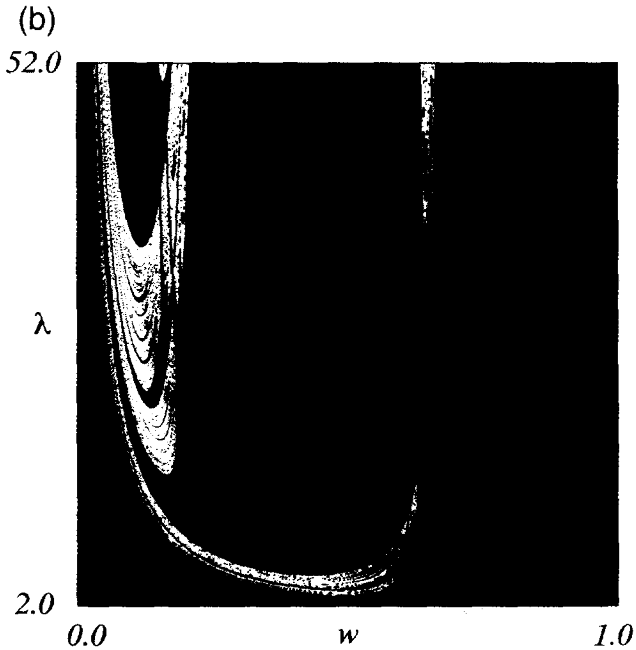


Fig. 6. a – Chaos Plot for cobweb model (8) with a = 0.8, b = 0.25. The black area is the set of parameters for which model (8) has a chaotic attractor. In this part of the parameter space, approximately 10% of the parameters are chaotic parameters. b – Iso-Period Plot for cobweb model (8) with a = 0.8. b = 0.25. The color coding is the same as in Fig. 2b. c – Blow-up of Fig. 6b. White shading corresponds to parameters leading to chaotic behavior. Many similarly looking parabolic-like structures ("shrimps") are embedded in the white region. Similar colors represent similar periods. For example, the light-blue region indicates parameters producing period-3 dynamics. Adjacent to the color corresponding to period m, one sees additional colors which indicate the location of the $m \times 2^n$ cascades. For example, the color black (adjacent to the period-3 region) represents period 6, etc.



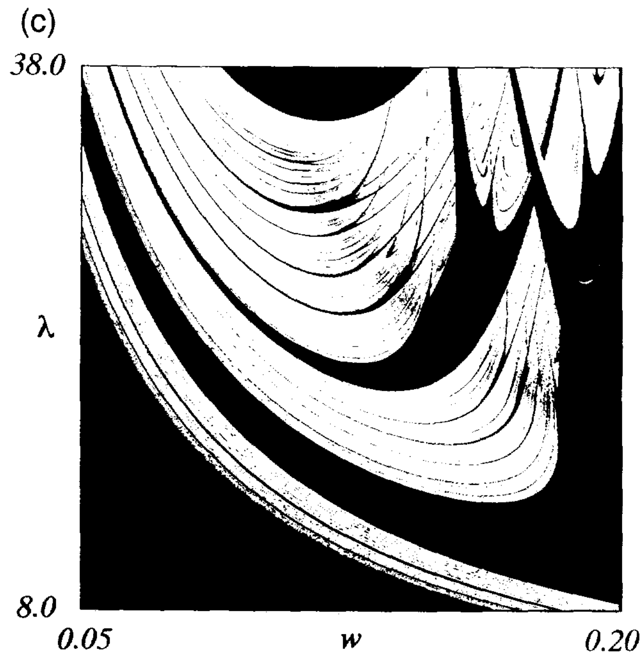


Fig. 6 (continued).

(a_0,b_0,λ_0,w_0) may dramatically differ from the dynamics for slightly different parameter values. In the parts of the parameter space shown in the figures, there are a number of places where the situation is rather complicate. On the other hand, period-m behavior (say, for example, m = 1, 2, 3, 4, 5) is prevalent. Fig. 6c shows with more detail the region where one observes the shrimps in parameter space. All in all, this shows *ipso facto* the richness of dynamical behavior seen when, in addition to considering the variation of the parameters involved, one exploits the consequences of simultaneously varying more than one parameter.

5. Conclusion

By explicitly considering the influence of simultaneously varying two or more parameters for the cobweb model with adaptive expectation, we showed that there are a number of features that are to be expected to be observed, i.e., measured, in the variable. We used reliable numerical methods as a tool to investigate the periodic (Iso-Period Plots) versus chaotic behavior (Chaos Plots) in the dynamics of this model. Colored regions in the parameter space indicate periodic behavior in the dynamics of the cobweb model, and the white regions indicate chaotic behavior. For creating Chaos Plots and Iso-Period Plots, one generally needs a lot of initial conditions to be tested. For the cobweb model with adaptive expectation, we showed that only two initial conditions have to be tested. We find the appearance of fractal structures in parameter space indicating qualitatively different dynamics when the parameters are perturbed slightly. Generally, by choosing appropriate paths in the parameter space, one may see different routes to chaos. The resulting pictures imply that periodic behavior is more prevalent than chaotic behavior in the dynamics of the cobweb model with adaptive expectation. The used plotting methods can be applied to parameter spaces of other economic models equally well. The methods that are presented in this paper, may be considered as a tool to investigate the dynamical behavior of a variety of economic models. Many (unobserved) phenomena are waiting to be discovered.

6. For further reading

Chiarella (1988) and Day and Hanson (1991).

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