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Impact of bistability in the synchronization of chaotic maps with delayed coupling and complex topologies

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Abstract

We investigate the impact of bistability in the emergence of synchronization in networks of chaotic maps with delayed coupling. The existence of a single finite attractor of the uncoupled map is found to be responsible for the emergence of synchronization. No synchronization is observed when the local dynamics has two competing chaotic attractors whose orbits are dense on the same interval. This result is robust for regular networks with variable ranges of interaction and for more complex topologies.

Keywords: Time-delay; Coupled maps; Synchronization; Complex networks

Synchronization of chaotic systems has been frequently studied nowadays because of its relevance in many situations, e.g. in nonlinear optics and fluid dynamics [1]. Since a realistic coupling between chaotic systems should consider finite-time propagation of the interaction, particular attention has been recently given to time-delayed coupling [2,3]. For instance, delayed coupling was used to study the emergence of multistability in noisy bistable elements [4] and to analyze anticipating synchronization of two excitable systems [5]. It is already known that for continuous systems the stability conditions of coupled elements is not influenced by the time-delay of the coupling when the elements are randomly coupled [6], and that synchronization is independent of the topology but depends on the average number of neighbors [3]. For discrete systems modeled by coupled map lattices, it was found that even with random delays, for adequate coupling strength an array of chaotic logistic maps is able to synchronize [3].

Here, we present robust numerical evidence that synchronization effects observed in lattices of coupled logistic maps are strongly dependent of the local map. Specifically, we show that synchronization may be easily destroyed by simply substituting the logistic map by a bistable map with two competing chaotic attractors. This is true both for regular and for complex networks.

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Our model consists of the usual array of coupled chaotic maps defined by

$$x_{t+1}(i) = (1-\varepsilon)f(x_t(i)) + \frac{\varepsilon}{\sum_{j=1}^{N}\eta_{ij}} \sum_{j \neq i} \eta_{ij}\omega_{ij} f(x_{t-\tau}(j)), \tag{1}$$

where τ represents the delay, $f(x_t)$ denotes the local map, ε is the coupling strength and N is the total number of nodes. The adjacency matrix $\{\eta_{ij}\}$ and the coupling strength ω_{ij} between nodes i and j depend on the network topology and for all i one has $\sum_{j=1}^{N}\omega_{ij} = N-1$. As usual, synchronized solutions are detected when the standard deviation of the amplitudes x_t is numerically zero, i.e., when $\sigma^2(t) = (1/(N-1))\sum_j (x_t(j) - \langle x_t \rangle)^2 \lesssim 10^{-20}$.

In this framework, we contrast the properties of the standard logistic map $f_l(x_t) = a - x_t^2$ characterized by a single finite attractor, with those found for the quartic map $f_q(x_t) = a - (a - x_t^2)^2$ having two stable finite attractors (bistability). Although the bifurcation diagram of these maps look numerically the same (see Fig. 1), at $a = 0.75$ the logistic map suffers a period-doubling bifurcation while the quartic map shows a tangent bifurcation where the fixed point splits into a *pair* of fixed points, each one with its own basin of attraction and doubling cascade. By contrast with the logistic map, the quartic map has two stable finite attractors beyond $a > 0.75$. In particular, in the fully chaotic regime ($a = 2$) the two chaotic orbits of the quartic map are dense in the interval $[-2, 2]$, i.e., any subinterval intersects both basins of attraction. Therefore, one could expect that for $a = 2$ the two maps are numerically indistinguishable when introduced in Eq. (1). However, as we show below, this is not the case.

We start by considering a regular network where each node i is symmetrically coupled to its neighbors, labeled as j, in both short- and long-range interaction regimes. For short-range coupling, the coupling strength decreases exponentially with the distance $|i-j|_{\text{mod}\,N/2}$, i.e., $\omega_{ij} \propto \exp(-\alpha|i-j|)$, while for long-range coupling the decrease is polynomial, $\omega_{ij} \propto |i-j|^{-\alpha}$. In both cases, α controls the range of interaction: $\alpha = 0$ yields the fully and uniformly connected network while for $\alpha \to \infty$ maps are decoupled.

Full synchronization is observed in the black regions in Fig. 2. Exponential coupling using logistic maps is illustrated in Figs. 2(a) and (b), while polynomial coupling using logistic and quartic maps is shown in Figs. 2(c) and (d) and (e) and (f), respectively. For quartic maps with exponential coupling the results are independent of τ and are similar to those of Fig. 2(a). For other even and odd values of τ, one obtains plots similar to the ones for $\tau = 6$ and 9, respectively. From Fig. 2 one clearly sees that, while for exponential coupling synchronization is observed only in a neighborhood of $\alpha = 0$, for polynomial coupling, the synchronization region spreads in a wider range of α-values, since the coupling strength decreases more slowly with the distance. Moreover, there is different behavior for the two types of local dynamics: for the logistic map there is a synchronization region within $0.1 \lesssim \varepsilon \lesssim 0.2$ while for the quartic map there is not. In the particular case of nearest-neighbors coupling this is almost the unique synchronization region corroborating recent findings [2].

For complex topologies bistability also suppresses the emergence of synchronized solutions as shown in Fig. 3. In order to ascertain the robustness of our results, we consider three different types of complex

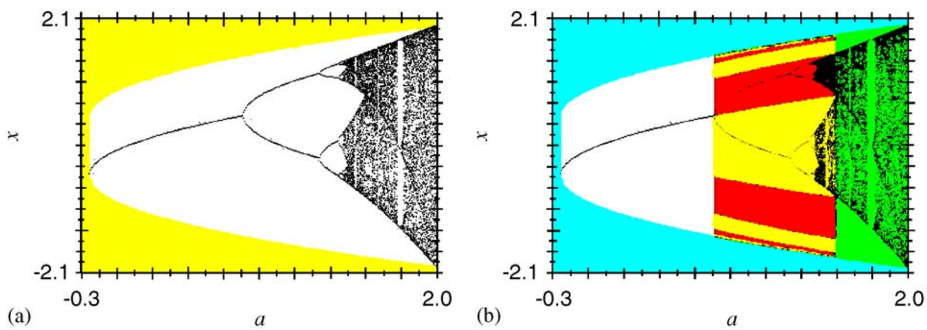

Fig. 1. Bifurcation diagrams of (a) the logistic map $x_{t+1} = a - x_t^2$ with one single finite stable attractor, and of (b) the quartic map $x_{t+1} = a - (a - x_t^2)^2$, where for $a > 0.75$ two stable attractors coexist. Different basins of attraction are represented with different tonalities.

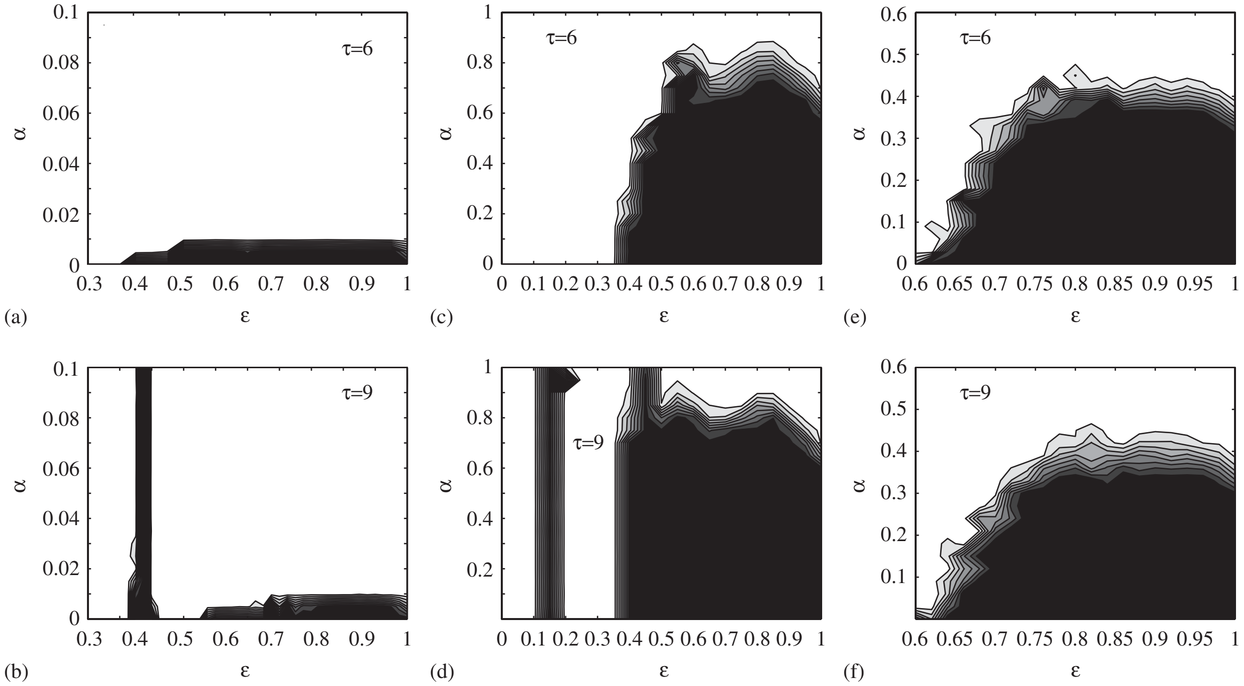

Fig. 2. Regions of full-synchronization (black) for regular networks with (a) and (b) exponential coupling using logistic maps, (c) and (d) polynomial coupling with logistic maps and (e) and (f) polynomial coupling with quartic maps. Here $a = 2$, $N = 10^3$, transients are 10^4, samples contain 50 initial configurations. τ is the delay defined in Eq. (1).

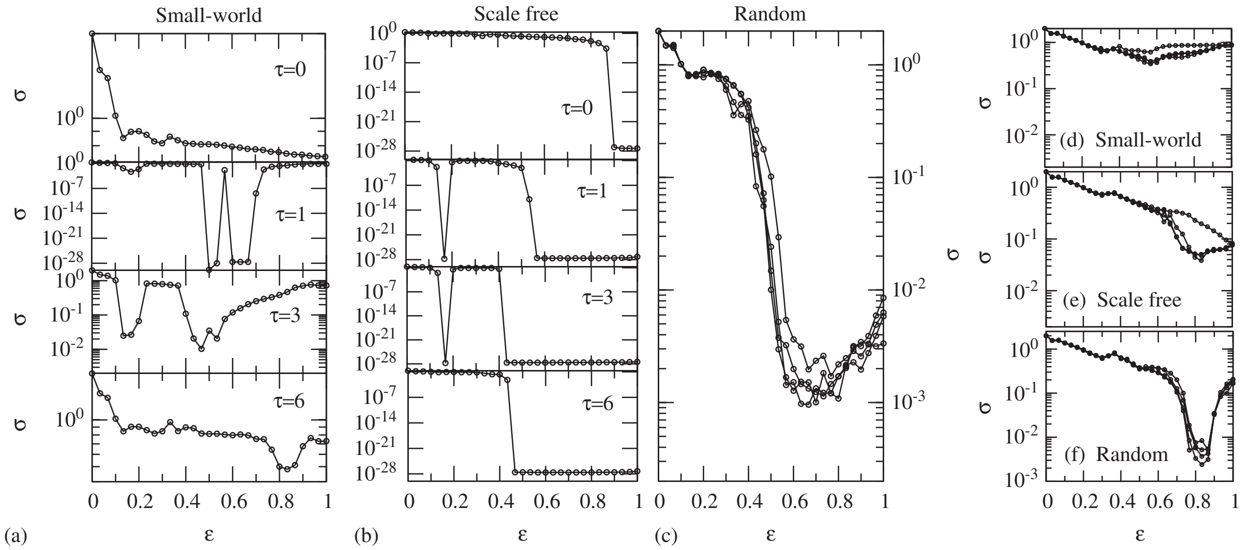

Fig. 3. Standard deviation σ of the node amplitudes as a function of the coupling strength in complex networks using (a–c) the logistic map and (d–f) the quartic map. In all cases $\langle k \rangle = 10$ and the conditions of Fig. 2 were used. The small-world network is constructed using the algorithm in [7] with a probability $p = 0.05$ for acquiring random connections, while the scale-free network is constructed from the Albert–Barabási model.

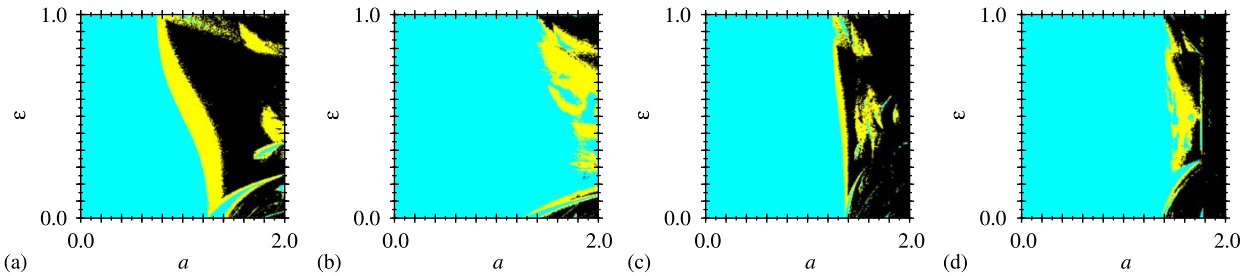

Fig. 4. Illustration of periodic (gray), chaotic (white) and hyperchaotic (black) regions of the synchronized solutions in parameter space (a and ε). Logistic map: (a) $\tau = 1$ and (b) $\tau = 2$. Quartic map: (c) $\tau = 1$ and (d) $\tau = 2$. For each pair of values, Lyapunov exponents were computed from intervals of 10^3 time-steps after discarding transients of 10^3 time-steps.

topologies, namely a random topology, a small-world topology and a scale-free network. Figs. 3(a–c) show the results for the logistic map, while Figs. 3(d–f) show the results for the quartic map. In all cases of Fig. 3 the average degree is fixed at $\langle k \rangle = 10$ neighbors, and we have taken, for simplicity, $\omega_{ij} = 1 - \delta_{ij}$ (uniform coupling).

In addition to suppression of synchronization in the case of the quartic map, one observes, for the particular case of scale-free networks of logistic maps, a very abrupt transition to synchronization. This transition, already reported in previous studies [8], is still not fully understood. Our simulations indicate that the coupling strength threshold is independent of the network size.

For all the cases above, the synchronized solution of Eq. (1) evolves according to $X_{t+1} = (1-\varepsilon)f(X_t) + \varepsilon f(X_{t-\tau})$, involving the two variables X_t and $X_{t-\tau}$. Therefore, the corresponding Jacobian yields two different Lyapunov exponents: if both exponents are negative the orbit is periodic, if one of the exponents is positive the orbit is chaotic and when both are positive the orbit is hyperchaotic. Figs. 4(a) and (b) show where each of the three regimes occurs for the logistic map either with even and odd time delays while Figs. 4(c) and (d) illustrate the same situations for the quartic map. Clearly, for $a = 2$ the quartic map shows hyperchaos in all the range of ε values, while in the logistic case only certain ε ranges exhibit hyperchaos. This fact could be related with the significant suppression of synchronized solutions when bistability sets in. For other odd or even time delays similar plots are obtained.

In conclusion, we found that bistable local dynamics in networks of chaotic maps with delayed coupling destroys synchronization for a wide variety of situations. In the literature, synchronization in networks with delayed coupling was only studied so far under the very particular unimodal local dynamics. As shown here, however, care should be taken when interpreting results obtained with logistic maps because synchronized solutions are too dependent on the maps ruling the dynamics.

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