ELSEVIER

Physica A 283 (2000) 17–23

PHYSICA A

www.elsevier.com/locate/physa

On the origin of periodicity in dynamical systems

Jason A.C. Gallas[a,b,c,*]

[a] *Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, 1749-016 Lisboa, Portugal*
[b] *Institut für Computer Anwendungen, Universität Stuttgart, Pfaffenwaldring 27, D-70569 Stuttgart, Germany*
[c] *Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brazil*

Abstract

We prove a theorem establishing a direct link between macroscopically observed periodic motions and certain subsets of intrinsically discrete orbits which are selected naturally by the dynamics from the skeleton of unstable periodic orbits (UPOs) underlying classical and quantum dynamics. As a simple illustration, an *infinite* set of UPOs of the quadratic (logistic) map is used to build ab initio the familiar trigonometric and hyperbolic functions and to show that they are just the first members of an infinite hierarchy of functions supported by the UPOs. Although all microscopic periodicities of the skeleton involve integer (discrete) periods only, the macroscopic functions resulting from them have real (non-discrete) periods proportional to very complicate non-integer numbers, e.g. 2π and $2\pi i$, where $i = (-1)^{1/2}$. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 05.45.−a; 03.20.+i; 02.10.Nj

Keywords: Periodicity; Quadratic map; UPOs; Periodic functions; Number theory

The aim of this paper is to investigate conditions which allow simple dynamical systems to support periodic behaviors. This question is of great importance due to the pervasive nature of the infinite set of unstable periodic orbits (UPOs) which is well known to form the skeleton underlying dynamical systems [1]. The possibility of using the set of periodic orbits to obtain very good semiclassical approximations of the atomic spectra of simple atoms leads us to wonder whether it is possible to find a direct link between the infinite set of UPOs and the *stable* periodicities (or aperiodicities) which surface macroscopically in physical systems. Or, equivalently, whether it is possible to understand the origin of stable periodicities from properties

* Correspondence address: Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970, Porto Alegre, Brazil.
E-mail address: jgallas@if.ufrgs.br (J.A.C. Gallas)

PII: S0378-4371(00)00123-0

of the skeleton of UPOs. This paper reports a result that casts some light onto this difficult question and shows that such a link indeed exists sometimes. An additional motivation is the fact that although the concept of *periodicity* is intuitive and clear to everyone, there is so far a quite surprising absolute lack of understanding about why some functions are able to display periodic behaviors while others are not. Thus, we ask: what sort of mathematical principles are responsible for the *genesis* of periodicities in arbitrary functions represented by, say, series expansions? What sort of generic "motor" is needed to transform an arbitrary power series into a periodic function?

For a wide class of dynamical systems, we prove a theorem showing that macroscopically measured periodicities, i.e., the periodicities of generic analytical functions representing solutions of equations of motion, originate from an infinite set of intrinsically discrete microscopic periodicities that are interconnected by subtle cooperative phenomena among subsets of orbital points (zeros) of the equations of motion. Using this theorem, we obtain two very general formulas, Eqs. (7) and (9), establishing macroscopic periodicities directly from the orbital points in phase space. As a simple application, the two formulas are used to build ab initio the familiar trigonometric and hyperbolic periodic functions, i.e., to build the simplest examples of periodic behaviors from a particular hierarchy of UPOs of the quadratic (logistic) equation.

To start, assume a generic function $f(z)$, thought as solution of an equation of motion of interest, represented formally by a power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \,, \tag{1}$$

where the coefficients c_k contain all physical ("control") parameters ruling the dynamics. Eq. (1) is quite general, being able to represent a full spectrum of dynamical behaviors, periodic or not. Substitution of Eq. (1) into equations of motion leads to very strong restrictions among the coefficients of $f(z)$, forcing them to be interconnected in a quite regular way. This interconnection is the object of our investigation.

Let $f^n(z)$ denote the nth composition of $f(z)$ with itself: $f^n(z) = f(f^{n-1}(z))$ with $f^1(z) = f(z)$. Then, our main result may be stated as follows.

Theorem. *If for an arbitrary function* $f(z)$ *there is an associated "ghost" function* $g(z)$, *analytic and non-constant, such that* $f[g(z)] = g(\theta z)$, *where* θ *is a nth root of unity,* $\theta^n = 1$, *then* $f^n(z) = z$.

Proof.

$$\begin{aligned}
&f[g(z)] = g(\theta z)\,,\\
&f^2[g(z)] = f[g(\theta z)] = g(\theta^2 z)\,,\\
&f^3[g(z)] = f[g(\theta^2 z)] = g(\theta^3 z)\,,\\
&\vdots\\
&f^n[g(z)] = f[g(\theta^{n-1} z)] = g(\theta^n z) = g(z)\,. \qquad \square
\end{aligned}$$

Corollary.

$$\sum_{j=1}^{n} f^j[g(z)] = \sum_{j=1}^{n} g(\theta^j z)\,. \tag{2}$$

The converse of this theorem, that $f^n(z) = z$ implies the existence of $g(z)$, is also true but the proof is less simple by far. We believe the theorem above to be responsible for the origin of periodic motions observed in dynamical systems and the remainder of the paper is dedicated to illustrate an application of the theorem to a generic algebraic dynamical system, the paradigmatic quadratic ("logistic") map.

Each individual n-periodic orbit forming the skeleton of an algebraic system must obey a polynomial equation, that we write generically as

$$z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n = 0\,. \tag{3}$$

To know the full skeleton means to know all zeros of these polynomials. So, for each polynomial we build the well-known sequence of sums s_m of the powers $m = 0, 1, 2, \ldots, k$ of its zeros [2,3].[1] Adding the set of k equations obtained by substituting each individual zero in Eq. (3), one easily sees that the sums s_m satisfy

$$s_k + c_1 s_{k-1} + c_2 s_{k-2} + \cdots + c_{k-1} s_1 + c_k\, k = 0\,. \tag{4}$$

With no loss of generality, we now focus the discussion on the simplest possible system, $z^n - 1 = 0$, for which Eq. (3) contains the least possible number of terms. Curiously, maximal simplification of Eq. (3) is achieved not for an artificially constructed example but for the most frequently studied system of all: the quadratic map $f(z) = a + z^2$. For $a = 0$, the cyclotomic case [4,5], we have the usual fixed point $z = 0$, with the remaining skeleton of UPOs, defined by the zeros of $z^n - 1 = 0$, $n = 1, 2, \ldots, \infty$, covering the unit circle in the complex plane. The polynomials $z^n - 1 = 0$ are the well-known cyclotomic polynomials studied by Gauss, among others. All orbital points are roots of unity. More general situations are discussed in Refs. [4,5].

For Eq. (3) to represent every individual n-periodic motion $z^n - 1 = 0$ we must fix $c_n = -1$ and $c_i = 0$ for $i = 1, 2, \ldots, (n-1)$, implying, by Eq. (4), $s_k = 0$ for $k < n$, $s_k = n$ for $k = n$. But a crucial property of the s_k is the fact that $s_k = s_{k-n}$ for $k > n$, i.e., *the "microscopic" sums s_k built with the zeros of the equations of motion are themselves periodic functions.* Thus, representing the n zeros of $z^n - 1 = 0$ by $\theta, \theta^2, \theta^3, \ldots, \theta^n = 1$ as usual and calling $s_m = \theta^m + \theta^{2m} + \theta^{3m} + \theta^{4m} + \cdots + \theta^{nm}$, one finds

$$s_m = \begin{cases} n & \text{if } m \text{ is multiple of } n\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5}$$

which is the central gear of the "motor" generating periodicities.

[1] According to Ref. [2] the sums s_n originate with A. Girard (1595–1632) and I. Newton (1642–1727). See also [3].

Now we turn to the function $g(z)$. Since nothing is known about this function, we start assuming it to be given as a generic power series

$$g(z) = \sum_{j=0}^{\infty} \frac{g_j}{j!} z^j , \tag{6}$$

where the coefficients are arbitrary. From Eq. (6) we easily obtain expansions for all $g(\theta^j z)$ appearing in Eq. (2) and, from them, after suitable multiplication by powers of θ and simplifications using Eq. (5) as often as necessary, we build the following sums:

$$\begin{aligned}
&g(\theta z) + g(\theta^2 z) + g(\theta^3 z) + \cdots + g(\theta^n z) \\
&\quad = n \left[g_0 + \frac{g_n}{n!} z^n + \frac{g_{2n}}{(2n)!} z^{2n} + \frac{g_{3n}}{(3n)!} z^{3n} + \cdots \right], \\
&\theta^{n-1} g(\theta z) + \theta^{n-2} g(\theta^2 z) + \theta^{n-3} g(\theta^3 z) + \cdots + g(\theta^n z) \\
&\quad = n \left[g_1 z + \frac{g_{n+1}}{(n+1)!} z^{n+1} + \frac{g_{2n+1}}{(2n+1)!} z^{2n+1} + \frac{g_{3n+1}}{(3n+1)!} z^{3n+1} + \cdots \right], \\
&\theta^{2(n-1)} g(\theta z) + \theta^{2(n-2)} g(\theta^2 z) + \theta^{2(n-3)} g(\theta^3 z) + \cdots + g(\theta^n z) \\
&\quad = n \left[\frac{g_2}{2!} z^2 + \frac{g_{n+2}}{(n+2)!} z^{n+2} + \frac{g_{2n+2}}{(2n+2)!} z^{2n+2} + \frac{g_{3n+2}}{(3n+2)!} z^{3n+2} + \cdots \right].
\end{aligned}$$

From the expressions on the left-hand side, one recognizes what the sum in Eq. (2) and similar ones are in fact doing: they are selecting equidistant coefficients from Eq. (6), thereby manifesting macroscopically the microscopic periodicity of Eq. (5).

Continuing the process one arrives at the generic *sum* or, equivalently, *average* of functions

$$\begin{aligned}
P_{n,m}(z) &= \frac{1}{n} [p_{n,m}(z)] \\
&= \frac{1}{n} [\theta^{m(n-1)} g(\theta z) + \theta^{m(n-2)} g(\theta^2 z) + \theta^{m(n-3)} g(\theta^3 z) + \cdots + g(\theta^n z)] \\
&= \frac{g_m}{m!} z^m + \frac{g_{n+m}}{(n+m)!} z^{n+m} + \frac{g_{2n+m}}{(2n+m)!} z^{2n+m} + \cdots ,
\end{aligned} \tag{7}$$

where the average is a linear combination of n terms obtained for $0 \leqslant m \leqslant n-1$. Notice that since for every n the numerical value of θ is fixed by the requirement $\theta^n = 1$, the sum $P_{n,m}(z)$ does not in fact depend on θ.

From Eq. (7), one may derive an additional equation by replacing z with ωz:

$$\begin{aligned}
&\frac{1}{n\omega^m} [\theta^{m(n-1)} g(\theta \omega z) + \theta^{m(n-2)} g(\theta^2 \omega z) + \cdots + g(\theta^n \omega z)] \\
&\quad = \frac{g_m}{m!} z^m + \frac{g_{n+m}\omega^n}{(n+m)!} z^{n+m} + \frac{g_{2n+m}\omega^{2n}}{(2n+m)!} z^{2n+m} + \cdots ,
\end{aligned} \tag{8}$$

the average involving *linear* combinations of the n sums, for $0 \leqslant m \leqslant n-1$.

By suitably choosing ω one may derive a variety of useful formulas. For example, take ω as a zero of the dual equation of motion $z^n + 1 = 0$. Then, clearly, $\omega^n = -1$, $\omega^{2n} = +1$, $\omega^{3n} = -1$, $\omega^{4n} = +1, \ldots$, a series that alternates signs periodically indefinitely, yielding an additional sum *dual* to $P_{n,m}(z)$:

$$\begin{aligned} Q_{n,m}(z) &= \frac{1}{n\omega^m}[q_{n,m}(z)] \\ &= \frac{1}{n\omega^m}[\theta^{m(n-1)} g(\theta\omega z) + \theta^{m(n-2)} g(\theta^2 \omega z) + \cdots + g(\theta^n \omega z)] \\ &= \frac{g_m}{m!} z^m + \frac{g_{n+m}\omega^n}{(n+m)!} z^{n+m} + \frac{g_{2n+m}\omega^{2n}}{(2n+m)!} z^{2n+m} + \cdots . \end{aligned} \tag{9}$$

The functions $P_{n,m}(z)$ and $Q_{n,m}(z)$ establish the link between the infinite UPOs and functions which are measurable macroscopically [4].[2]

Now, we apply Eqs. (7) and (9) to build very familiar examples of periodic functions, namely, trigonometric and hyperbolic functions. To this end, consider the infinite series

$$g(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots . \tag{10}$$

In this case, $P_{n,m}(z)$ and $Q_{n,m}(z)$ simplify considerably

$$\begin{aligned} P_{n,m}(z) &= \frac{1}{n}[\theta^{m(n-1)} e^{\theta z} + \theta^{m(n-2)} e^{\theta^2 z} + \theta^{m(n-3)} e^{\theta^3 z} + \cdots + e^{\theta^n z}] \\ &= \frac{z^m}{m!} + \frac{z^{n+m}}{(n+m)!} + \frac{z^{2n+m}}{(2n+m)!} + \cdots , \end{aligned}$$

$$\begin{aligned} Q_{n,m}(z) &= \frac{1}{n\omega^m}[\theta^{m(n-1)} e^{\theta\omega z} + \theta^{m(n-2)} e^{\theta^2 \omega z} + \theta^{m(n-3)} e^{\theta^3 \omega z} + \cdots + e^{\theta^n \omega z}] \\ &= \frac{z^m}{m!} + \frac{\omega^n z^{n+m}}{(n+m)!} + \frac{\omega^{2n} z^{2n+m}}{(2n+m)!} + \cdots . \end{aligned}$$

For $n = 1$ we may only have $m = 0$. In addition, $\theta = 1$, thus implying $\theta^j = 1$ for all $j = 1, 2, \ldots$. Similarly, $\omega = -1$, implying $\omega^{2j} = 1$ and $\omega^{2j+1} = -1$ for $j = 1, 2, \ldots$. Therefore,

$$P_{1,0}(z) = e^z = 1 + z + \frac{z^2}{2!} + \cdots, \quad Q_{1,0}(z) = e^{-z} = 1 - z + \frac{z^2}{2!} - \cdots .$$

These two functions form the standard basis for solving linear first-order differential equations. They are both periodic, with purely imaginary periods. Although we started from $g(z) = e^z$, the motor of periodicity generated automatically its dual e^{-z}, providing, therefore, a complete basis for solving first-order differential equations.

[2] Technically, an n-adic arithmetic emerges as a *natural consequence* of the underlying periodicity of $g(z)$, without any need for introducing the concept of "congruence" somewhat artificially. Thus, the dynamics automatically *generates* congruences in profusion.

For $n=2$ we have $\theta=-1$ and, correspondingly,

$$P_{2,0}(z)=\frac{e^{-z}+e^{z}}{2}=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots=\cosh(z)\,,$$

$$P_{2,1}(z)=\frac{-e^{-z}+e^{z}}{2}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots=\sinh(z)\,.$$

Similarly, with $\omega=(-1)^{1/2}=i$, one finds

$$Q_{2,0}(z)=\frac{e^{-iz}+e^{iz}}{2}=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots=\cos(z)\,,$$

$$Q_{2,1}(z)=\frac{-e^{-iz}+e^{iz}}{2i}=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots=\sin(z)\,,$$

expressions which recover familiar periodic functions, circular (having only *real* periodicities) and hyperbolic (having only *imaginary* periodicities), as indicated. These functions form a basis for second-order differential equations.

For $n=2$ the motor generated automatically not only the purely imaginary periodicities, that one could argue to be somewhat embedded in $g(z)$, but also additional "resonances" having real period, a remarkable accomplishment. As may be verified easily, starting with a series corresponding to a circular function leads automatically to the hyperbolic ones. Therefore, it is totally irrelevant whether one starts from real or complex periodicities: *given a single piece of information, the motor recovers automatically all other members in the same periodicity class*, consistently and systematically for $n=1,2,3,\ldots$. We searched the literature for ab initio derivations of trigonometric and periodic functions and although we found a few [6–11], we were not able to find the derivation above. At any rate, our objective is to call attention to the subtle action of the motor of periodicity, not to rederive known results.

For $n=3$ we have $\theta^3=1$ and $\theta=\frac{1}{2}(-1\pm i\sqrt{3})$, implying $\theta^2=\frac{1}{2}(-1\mp i\sqrt{3})$, which produces the functions $P_{3,0}(z)=(e^{\theta z}+e^{\theta^2 z}+e^{\theta^3 z})/3$, $P_{3,1}(z)=(\theta^2 e^{\theta z}+\theta e^{\theta^2 z}+e^{\theta^3 z})/3$, $P_{3,2}(z)=(\theta e^{\theta z}+\theta^2 e^{\theta^2 z}+e^{\theta^3 z})/3$. From $\omega^3=-1$ it follows that $\omega=-1$ or $\omega=\frac{1}{2}(1\mp i\sqrt{3})$ and $\omega^2=\frac{1}{2}(1\pm i\sqrt{3})$ yielding $Q_{3,0}(z)=(e^{-\theta z}+e^{-\theta^2 z}+e^{-\theta^3 z})/3$, $Q_{3,1}(z)=-(\theta^2 e^{-\theta z}+\theta e^{-\theta^2 z}+e^{-\theta^3 z})/3$, $Q_{3,2}(z)=(\theta e^{-\theta z}+\theta^2 e^{-\theta^2 z}+e^{-\theta^3 z})/3$. The following identities hold among the functions $P_{3,m}(z)$: $P''_{3,0}(z)=P'_{3,2}(z)=P_{3,1}(z)$, $P''_{3,1}(z)=P'_{3,0}(z)=P_{3,2}(z)$, $P''_{3,2}(z)=P'_{3,1}(z)=P_{3,0}(z)$, with analogous formulas existing for $Q_{3,m}(z)$. They form a convenient basis for solving differential equations of third order. From suitable sums of $\theta^m g(\theta^j z)$ one also obtains families of *addition* and *multiplication* formulas similar to the known results for $P_{2,m}$ (hyperbolic) and $Q_{2,m}$ (trigonometric) functions. But we will not go into this here. The new functions found for $n=3$ are of help in practical applications, e.g. in Ref. [12].

To conclude, we would like to point out an additional interesting consequence of the formulas above. Although the skeleton of UPOs involves an infinite quantity of periods characterized invariably by integer (discrete) numbers, the macroscopic periodicities obtained from them involved always multiples, real or complex, of $\pi=3.14\ldots$, a transcendental (non-integer) number. Thus, although the motor is microscopically driving the dynamics in a "quantized" (discrete) way, always pumping integer periods, the periodicities that surface macroscopically do not seem to be connected with integers in any direct and obvious way. Any attempt of explaining the fact that *both sets of periodicities lie mathematically in rather different number-fields* requires letting n to grow indefinitely, i.e., requires going from polynomial equations of motion to sets of entire functions, a quite delicate step [13,14].[3] Thus, Eqs. (7) and (9) confront us with a new interesting aspect of the problem: to understand the intricacies of the transition between periodicity and aperiodicity (chaos) we must disentangle properties which are number-theoretical from those depending on the nature of the equations of motion and understand the subtle interconnection between integer and non-integer periodicities.

I wish to acknowledge the opportunity to work in this field without distraction made possible by the kind invitation and hospitality of Prof. João Corte-Real, Universidade de Lisboa. The encouragement and fruitful discussions with Prof. Hans J. Herrmann, Universität Stuttgart, are also gratefully acknowledged. I am indebted to Prof. Guido A. Raggio, Universidade de Cordoba, Argentina, for pointing out an inconsistency in an earlier draft of this paper. This work was partially supported by the Program PRAXIS XXI (Portugal) and by CAPES and CNPq (Brazil).

References

[1] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York, 1990.
[2] K.Th. Vahlen, Encyklopädie der mathematischen Wissenschaften, Vol. I, paper I B3b, Teubner, Leipzig, 1898–1904, pp. 449–479.
[3] J.W. Archbold, Algebra, Pitman and Sons, London, 1958 (Chapter 11).
[4] J.A.C. Gallas, Europhys. Lett. 47 (1999) 649.
[5] J.A.C. Gallas, Infinite hierarchies of nonlinearly inherited periodic orbits, to appear.
[6] E.P. Wigner, Amer. Math. Monthly 59 (1952) 669.
[7] R. Grammel, Arch. Math. 1 (1948) 47.
[8] R. Grammel, Ing. Arch. 16 (1948) 188.
[9] R. Grammel, Ing. Arch. 18 (1950) 250.
[10] D. Shelupsky, Amer. Math. Monthly 66 (1959) 879.
[11] H. Kaufman, Amer. Math. Monthly 67 (1960) 753; and several references therein.
[12] H. Flyvbjerg, E. Jobs, S. Leibler, Kinetics of self-assembling microtubules: an "inverse problem" in biochemistry, Proc. Nat. Acad. Sci. USA 93 (1996) 5975.
[13] D. Hilbert, Über das Unendliche, Math. Ann. 95 (1926) 161 (French translation (by A. Weil): Sur l'infini, Acta Math. 48 (1926) 91.
[14] G. Gentzen, Die Widerspruchsfreiheit der reinen Zahlentheorie, Math. Ann. 112 (1936) 493.

[3] A detailed discussion of this delicate step was given by Hilbert [13], see also, Ref. [14].