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Period four stability and multistability domains for the Hénon map

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Abstract

We report the exact analytical expression of the surface $W_4(a, b; \lambda) = 0$ defining stability domains for period-4 motions in the Hénon map, valid for arbitrary eigenvalues λ and parameters a and b . For $\lambda = +1$ (fold bifurcations) the expression reproduces all previous results and gives a new one. For $\lambda = -1$ (flip bifurcations) it gives analytically the missing boundary needed for the rigorous delimitation of all period-4 stability domains and for the investigation of the arithmetic nature of parameters and trajectories. © 2001 Elsevier Science B.V. All rights reserved.

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An outstanding open problem in chaotic dynamics is determining accurately the location and extent of domains of stability for period k motions as well as determining the sequences of bifurcations that result in the creation of a Smale horseshoe in the system [1–4]. Despite the work done over the last 15 years or so, much still remains to be done. In this context, of particular interest are explicit analytical results to complement the impressive body of information already accumulated in so many theorems in the literature and by the plethora of very detailed numerical simulations.

Conceptually, the computation of stability domains in parameter space is a quite trivial task since what one needs to do is to simply eliminate the dynamical variables between (i) the equations of motion and (ii) the eigenvalue equation ruling the stability. Computationally, however, the difficulty lies in the fact that for nonlinear systems this task becomes impracticable very quickly. For the Hénon map, for instance, a brute-force computation of stability domains for periods higher than 3 is not yet feasible

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with present day commercial software and hardware. This computation, however, may be performed using special purpose routines developed to handle certain symmetries imposed by the equations of motion [5].

Our main purpose is to report analytical results obtained with the aforementioned routines for the paradigmatic Hénon map $(x, y) \mapsto (a - x^2 + by, x)$. More specifically, we report the exact analytical expression for the surface $W_4 \equiv W_4(a, b; \lambda)$ defined by all parameters (a, b) which lead to period-4 motions, valid for arbitrary eigenvalue λ .

As is known, bifurcations occur for nonhyperbolic periodic points, i.e., for points where at least one eigenvalue has modulus 1. At *folds*, when $\lambda = +1$, the particular cut $W_{k+} \equiv W_k(a, b; +1)$ reproduces all previous results [6–13], and gives a new one. For *flips*, $\lambda = -1$, $W_{k-} \equiv W_k(a, b; -1)$ yields an analytical expression for the missing boundary which allows the rigorous delimitation of all period-4 stability domains and the investigation of the arithmetic nature of boundary curves in parameter and in phase space. We omit derivations and write down the final result of a long computation:

$$W_4 = W_4(a, b; \lambda) = \sum_{i=0}^6 d_i a^i, \quad (1)$$

where $d_1 = 0$, $d_6 = -4096\lambda^3$, $d_5 = 4096\lambda^3(3b^2 + 2b + 3)$, and

$$d_4 = -256\lambda^2[(1 + 48\lambda)b^4 + 68\lambda b^3 + 38\lambda b^2 + 68\lambda b + (\lambda + 48)\lambda],$$

$$d_3 = 256\lambda^2[(1 + 48\lambda)b^6 + (6 + 68\lambda)b^5 + (\lambda^2 - 82\lambda)b^4 - 212\lambda b^3 + (\lambda^2 - 82\lambda)b^2 + (6\lambda^2 + 68\lambda)b + \lambda^2 + 48\lambda],$$

$$d_2 = -16\lambda[(32\lambda + 1)b^4 - 60\lambda b^3 - 186\lambda b^2 - 60\lambda b + \lambda^2 + 32\lambda] \\ \times [(16\lambda - 1)b^4 + 12\lambda b^3 - 6\lambda b^2 + 12\lambda b - \lambda^2 + 16\lambda],$$

$$d_0 = -[(16\lambda - 1)b^4 + 12\lambda b^3 - 6\lambda b^2 + 12\lambda b - \lambda^2 + 16\lambda]^3.$$

This surface has degrees 6, 12 and 6 in a , b and λ , respectively. From this expression it is possible to obtain now a number of interesting results.

Fig. 1 displays two curves in parameter space, solutions of $W_{4+} = 0$ and $W_{4-} = 0$. These curves delimit the stability domain for period-4 motions. From its shape one sees that, instead of a single domain, it is sometimes convenient to consider it as made of two distinct domains: one belonging to the period-doubling 2^ℓ cascade, the other corresponding to the birth of a ‘new’ period-4 stability island. The boundaries of this new island contain a much richer structure, broadening considerably near $b = -1$ and $+1$. Along $b = +1$ there is a cuspidal structure. By smoothly changing parameters along any loop enclosing this cuspidal structure it is possible to move continuously between different Riemann sheets, a simple consequence of the algebraic structuring of the orbital equations which manifests itself ‘macroscopically’ through a sort of hysteresis in the symbolic coding plane of the orbits. A similar phenomenon was observed numerically by Hansen [14] in a more complicated setting, involving circulation around a period-6 cuspidal structure.

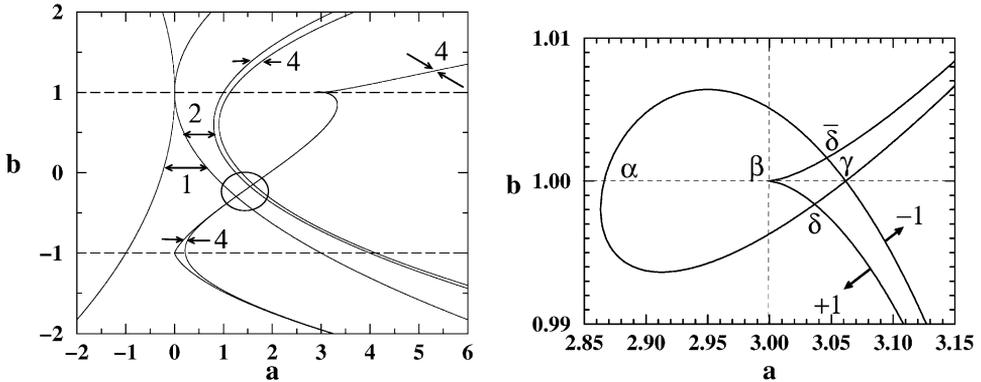


Fig. 1. *Left side*: The two period-4 stability domains of the Hénon map. For reference, stability domains for periods 1 and 2 are also indicated. *Right side*: The cuspidal structure existent along $b=1$. Smooth change of parameters along a loop enclosing this structure changes an orbit into another one, with different symbolic dynamics. See Ref. [5].

The $\lambda = +1$ boundary is particularly tame since W_{4+} may be readily decomposed into three factors, $W_{4+} \equiv w_{4+}^{(1)} w_{4+}^{(2)} w_{4+}^{(3)}$, where

$$w_{4+}^{(1)} = 4a - (5b^2 - 6b + 5) = 4a - 4(1 - b)^2 - (1 + b)^2, \tag{2}$$

$$\begin{aligned} w_{4+}^{(2)} &= 64a^3 - 144(1 + b)^2 a^2 + 108(1 + b)^4 a - 27(5 - 6b + 5b^2)(1 + b)^4, \\ &= [4a - 3(1 + b)^2]^3 - 27[2(1 - b)(1 + b)^2]^2, \end{aligned} \tag{3}$$

$$\begin{aligned} w_{4+}^{(3)} &= 16a^2 + 8(1 + b)^2 a + (5 - 6b + 5b^2)(1 + b)^2 \\ &= [4a + (1 + b)^2]^2 + 4(1 - b)^2(1 + b)^2. \end{aligned} \tag{4}$$

For $\lambda = -1$ no factorization is possible and one has to deal with Eq. (1) directly.

The expressions for $w_{4+}^{(1)}$ and $w_{4+}^{(2)}$ are well-known [6–13]. Except for $b = \pm 1$, all solutions of $w_{4+}^{(3)} = 0$ are complex: $4a = -(1 + b^2) \pm i2(1 - b^2)$. The factor $w_{4+}^{(3)}$ seems to be new. This factor is rather interesting because the lines $b = \pm 1$ correspond to the nondissipative (Hamiltonian) limits of the map. Solutions are degenerate and the dynamics of such complex, ghost [15,16], orbits is very rich but we will not go into this here.

From $w_{4+}^{(1)}$ and $w_{4+}^{(2)}$ we obtain two *conjugate* intersections $(a_D, b_D) = (-9\eta_1, \eta_1) \simeq (1.5441558, -0.1715728)$, and $(a_{\bar{D}}, b_{\bar{D}}) = (-9\eta_2, \eta_2) \simeq (52.4558441, -5.82842712)$, where $\eta_1 = -3 + 2\sqrt{2}$ and $\eta_2 = -3 - 2\sqrt{2}$. The last intersection is physically meaningless since it lies outside the interval $-1 \leq b \leq 1$. Notice that $\eta_2 = 1/\eta_1$, i.e., they are *reciprocal numbers*. To each physical solution b there is a corresponding unphysical conjugate $1/b$, and vice-versa. Moreover, since $w_{4+}^{(1)}$ and $w_{4+}^{(2)}$ are factors of W_{4+} , one sees that both intersections are, in fact, self-intersections of W_{4+} .

From the coordinates involving η_1 and η_2 we recognize that *coordinates of intersection points are not algebraically independent of each other* [12,13]. In both cases, a and b are functions of an underlying ‘fundamental constant’, η_1 or η_2 . Both coordinates

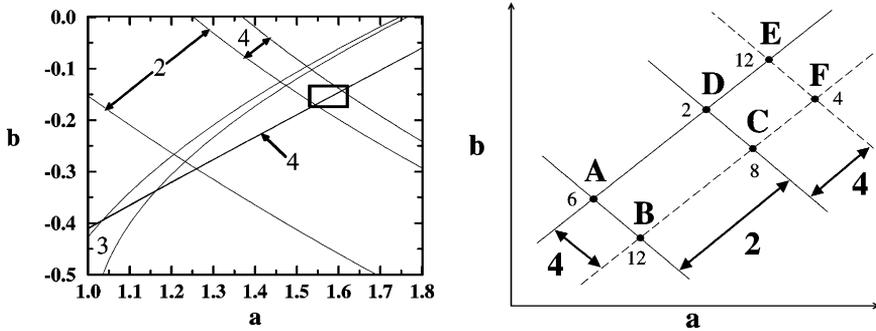


Fig. 2. *Left side*: The multistability domain defined by the intersection of both period-4 domains, inside the rectangle. *Right side*: Esquematic view of the multistability domain $CDEF$ where two different period-4 orbits coexist. Small numbers near each vertex refer to the degree of the algebraic numbers defining the vertex. Full (dashed) lines indicate a $\lambda = +1$ ($\lambda = -1$) boundary. Large numbers indicate the periodicity in the domain. Multistability also exists in $ABCD$, where periods 2 and 4 coexist.

are *quadratic numbers* [17] while their ratio $\kappa = a/b$ is an integer, i.e., an arithmetic quantity of degree half that of η_1 (or of η_2). Furthermore, both η_1 and η_2 , are *units* [17] in $\mathbb{Q}(\sqrt{2})$, since their product (norm) is 1. These arithmetical properties are very helpful to locate interesting behaviors in phase-space [12,13].

Particularly noteworthy in Fig. 1 are the points β and γ . The first corresponds to a discontinuity in the derivative of the $\lambda = +1$ locus, while at γ there is a *self-intersection* of the $\lambda = -1$ locus. The $\lambda = -1$ locus intersects $b = 1$ at the zeros of $4a^2 - 16a + 13 = 0$ and $4a^2 - 8a - 13 = 0$. The zeros corresponding to α and γ are $(a_\alpha, b_\alpha) = (2 + \sqrt{3}/2, 1) \simeq (2.86602540, 1)$, and $(a_\gamma, b_\gamma) = (1 + \sqrt{17}/2, 1) \simeq (3.06155281, 1)$. The $\lambda = +1$ locus intersects $b = 1$ at the zeros of $(a - 3)^3(a - 1)(a + 1)^2 = 0$. Points δ and $\bar{\delta}$ seen in Fig. 1 are located at $(a_\delta, b_\delta) \simeq (3.03646342, 0.998381041)$ and $(a_{\bar{\delta}}, b_{\bar{\delta}}) \simeq (3.04631917, 1.001621584)$, and depend on zeros of the reciprocal polynomial

$$6749(b^{12} + 1) + 66096(b^{11} + b) + 137160(b^{10} + b^2) - 7776(b^9 + b^3) - 228585(b^8 + b^4) - 58320(b^7 + b^5) + 169344b^6 = 0. \tag{5}$$

Fig. 2 shows the domain $CDEF$ where both period-4 orbits coexist. Point D was already defined. Another particularly interesting boundary point is the $\lambda = -1$ self-intersection vertex F located at $(a_F, b_F) = (-(87 + 20\sqrt{399})b^*/15, b^*) \simeq (1.60817868, -0.14453378)$, where the ‘fundamental quantity’ is $b^* = -(\xi - (\xi^2 - 15^2)^{1/2})/15$, where $\xi = 33 + \sqrt{399}$. The minimal polynomials [17] are now $225a^4 - 17868a^3 + 42442a^2 - 26860a + 6241 = 0$, and $15b^4 + 132b^3 + 214b^2 + 132b + 15 = 0$, the reciprocal one decomposing as $(b^2 - \sigma_+b + 1)(b^2 - \sigma_-b + 1) = 0$, where $\sigma_\pm = -(66 \pm 2\sqrt{399})/15$. This peculiar decomposition in reciprocal quadratic factors is a characteristic of the Hénon map.

The location of the remaining vertices C and E may be readily obtained from Eq. (1). Here, we give only the minimal polynomials for b and approximate values of the relevant zeros. The vertex E is located near $(a_E, b_E) = (1.60754787, -0.144189025)$, where

b_E is a zero of Eq. (5). The vertex C lies near $(a_C, b_C) = (1.54486140, -0.1719385)$, both coordinates now being functions of a zero of the reciprocal octic $p_8(b) = 261(b^8 + 1) + 648(b^7 + b) - 3372(b^6 + b^2) + 7896(b^5 + b^3) - 9710b^4$.

From a theoretical point of view, several interesting results emerge from Eq. (1) and from the exact expressions of intersection coordinates obtained after disentangling properties which are purely number-theoretical from those depending on the nature of the equations of motion. First, intersection *coordinates* (a, b) are *not independent of each other*: they are all rationally interconnected by the relation $a = \kappa b$ where, κ is an algebraic number of degree half that of b . Physically, the analogous situation in a phase diagram of water, say, would be to have temperature and pressure at the triple point not really independent of each other but depending both on a more fundamental quantity of pure arithmetical nature, a suitable numerical constant imposed by the number-field underlying the equation of state [15,16]. Second, *intersection coordinates are functions of very specific units* [17] in *characteristic number fields* enforced by the equations of motion, corroborating earlier findings [12,13]. Third, *b-coordinates of all intersection points are roots of reciprocal quadratic equations* of the form $b^2 - \sigma b + 1 = 0$. In other words, it is always possible to find suitable number-fields allowing one to factor the high-degree minimal polynomials defining b coordinates into a product of reciprocal quadratic factors. Physically, this means that to every physical intersection b_p lying in the interval $-1 < b < 1$ there is an associated unphysical conjugate intersection $b_u = 1/b_p$ lying outside this interval.

Having found $W_4(a, b; \lambda)$, an interesting question now is to investigate the parameter dependence of the family of quartics $x^4 - S(a, b)x^3 + U(a, b)x^2 - V(a, b)x + P(a, b) = 0$, constructed with points of period-4 orbits. A natural question is to inquire about the arithmetical signature of the number-fields involved in *orbits* involved in the nucleation of period-4 stability and the properties of the minimal polynomials in the domains of multistability. Do coexisting orbits share similar Galois group? Is it possible to find an ‘arithmetical equivalent’ of the symbolic dynamics underlying the construction of generating partitions [18–20]? If so, would it be applicable for arbitrary changes of parameters? We intend to address some of these questions in a future publication.

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References

- [1] P. Grassberger, H. Kantz, Phys. Lett. A 113 (1985) 235.
- [2] P. Grassberger, H. Kantz, U. Moening, J. Phys. A 22 (1989) 5217.
- [3] M.J. Davis, R.S. MacKay, A. Sannami, Physica D 52 (1991) 171.
- [4] D. Sterling, H.R. Dullin, J.D. Meiss, Physica D 134 (1999) 153.
- [5] A. Endler, J.A.C. Gallas, 2000, preprint.
- [6] H. El Hamouly, C. Mira, C. Rendus, Acad. Sc. Paris 293 (1981) 525.
- [7] H. El Hamouly, C. Mira, C. Rendus, Acad. Sc. Paris 294 (1982) 387.
- [8] C. Mira, Chaotic Dynamics, World Scientific, Singapore, 1987.
- [9] D.L. Hitzl, F. Zele, Physica D 14 (1985) 305.
- [10] H. Yung-nien, Chinese Phys. Lett. 2 (1985) 97.

- [11] H. Yung-nien, *Sci. Sinica* 29 (1986) 1302.
- [12] J.A.C. Gallas, *Physica A* 222 (1995) 125.
- [13] M.W. Beims, J.A.C. Gallas, *Physica A* 238 (1997) 225.
- [14] K.T. Hansen, *Phys. Lett. A* 165 (1992) 100.
- [15] J.A.C. Gallas, *Europhys. Lett.* 47 (1999) 649.
- [16] J.A.C. Gallas, Infinite hierarchies of nonlinearly dependent periodic orbits, *Phys. Rev. E* 63 (2001) 016216.
- [17] I.N. Stewart, D.O. Tall, *Algebraic Number Theory*, 2nd edition, Chapman & Hall, London, 1994.
- [18] F. Christiansen, A. Politi, *Physica D* 109 (1997) 32.
- [19] L. Jaeger, H. Kantz, *J. Phys. A* 30 (1997) L567.
- [20] R. Davidchak, Y.-C. Lai, E. Bollt, M. Dhamala, *Phys. Rev. Lett.* 84 (2000) 1353.