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Fractal and riddled basins: arithmetic signatures in the parameter space of two coupled quadratic maps

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Abstract

We investigate the parameter space of two coupled quadratic (logistic) maps. Of special interest is the analytical characterization of the precursors leading to riddled basins. We delimit stability domains for orbits with the two lowest periods. In addition, we study the singularities of the phase-space surfaces obtained by eliminating *all parameters* from the equations of motion.

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Riddled basins are a very intricate and relatively abundant type of fractal structures in phase-space of physical systems which attract enormous attention nowadays [1–5]. The sensitivity to initial conditions in phase-space with riddled basins is so extreme that no matter how small a volume is chosen, it will always contain initial conditions leading to different final states, implying unpredictability at all scales of resolution. Riddled basins were observed very early in lattices of coupled maps [6–8] and shown to arise when a chaotic motion is restricted to an invariant subspace of the total phase space [9]. Here we report an investigation of the parameter space of a simple system

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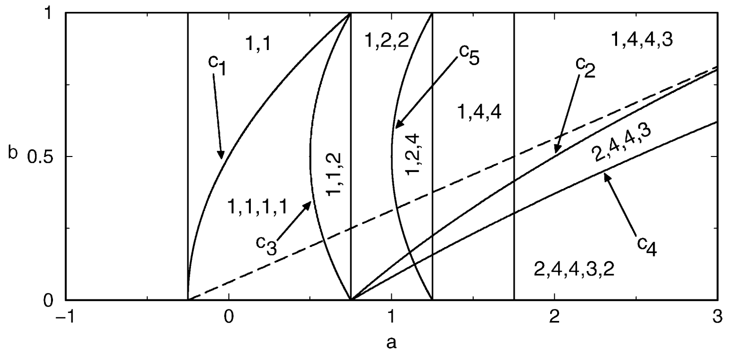

Fig. 1. Stability and multistability domains. Numbers like 2, 4, 4, 3 indicate coexistence of periods 2, 3 with two different orbits of period 4. The dashed line goes from $p_1 \equiv (-1/4, 0)$ to $p_2 \equiv (3, \sim 0.8)$.

composed of two coupled quadratic maps interconnected by linear terms:

$$x_{n+1} = a - x_n^2 + b(x_n - y_n) \quad \text{and} \quad y_{n+1} = a - y_n^2 + b(y_n - x_n)\,, \tag{1}$$

where a is the local parameter and b represents the coupling. Particular emphasis is given to the arithmetical structure (number fields [10,11]) defining parameter boundaries and vertices of stability domains of motions with low periods.

Algebraic equations of motion imply an infinite family of polynomials [12,13] whose zeros define the orbital points. Here, all orbital points with periods 1 and 2 are zeros of $P_1(x) \equiv p_1^{(1)}(x)p_1^{(2)}(x)$ and $P_2(x) \equiv P_1(x)\; p_2^{(1)}(x)p_2^{(2)}(x)p_2^{(3)}(x)$, where $p_1^{(1)}(x) = x^2 + x - a$, $p_1^{(2)}(x) = x^2 + (1-2b)x + 2b^2 - b - a$, $p_2^{(1)} = x^2 - x - a + 1$, $p_2^{(2)} = x^2 - (1 + 2b)x + (b+1)(1+2b) - a$, and $p_2^{(3)}$ is an octic polynomial. Following Ref. [14], we determined stability boundaries, parameterized by the eigenvalue λ. The boundaries of domains of lowest periodicity directly related to the $1 \to 2 \to 4 \to \cdots$ cascade are

$$W_1^+ = (4a+1)(4a-4b^2+1)^3, \tag{2}$$

$$W_1^- = (4a-3)(4a-4b^2-3-8b)(-3+4a+4b-4b^2)^2\,, \tag{3}$$

$$W_2^+ = (b-1)^8(4a-3)^2(-3+4a+4b-4b^2)^2 \times(4a-4b^2-3-8b)^2(4a-4b^2-3-12b)^2\,, \tag{4}$$

$$W_2^- = (b-1)^8(4a-5)^2(4a-4b^2+4b-5)^2 \times(25-40a+16a^2+100b-80ba+144b^2-32ab^2+80b^3+16b^4)^2, \tag{5}$$

where $+$ and $-$ refer to $\lambda = +1$ and -1, respectively. The W_2 surfaces contain additional factors of higher degrees which are not interesting here.

Fig. 1 shows the location of several stability domains for orbits of periods 1 and 2. Sequences of numbers indicate coexistence of the periods indicated by the numbers.

From the expression for P_1 one sees that there are four period-1 fixed points, some of them being real only to the right of the c_1 curve. The four vertical lines, for $a=-\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}$, correspond to the symmetric $x_t=y_t$ bifurcations of the quadratic map. Relevant boundaries are

$$c_1 = 4a - 4b^2 + 1 \quad \text{(Birth of period 1)}, \tag{6}$$

$$c_2 = 4a - 4b^2 - 3 - 8b \quad \text{(Bifurcation } 1\to 2\text{)}, \tag{7}$$

$$c_3 = 4a - 4b^2 - 3 + 4b \quad \text{(Bifurcation } 1\to 2\text{)}, \tag{8}$$

$$c_4 = 4a - 4b^2 - 3 - 12b \quad \text{(Birth of a new period 2)}, \tag{9}$$

$$c_5 = 4a - 4b^2 - 5 + 4b \quad \text{(Bifurcation } 2\to 4\text{)}. \tag{10}$$

The curves c_2 and c_5 intersect at $(a,b)=(\frac{10}{9},\frac{1}{6})$ while c_4 and c_5 intersect at $(\frac{65}{64},\frac{1}{8})$. Notice that c_2, c_4 cross the line $a=\frac{5}{4}$ for $b=-1+\sqrt{6}/2$ and $b=-\frac{3}{2}+\sqrt{11}/2$.

If instead of eliminating x and y from the equations of motion we eliminate both parameters, a and b, we obtain high-degree surfaces which, among others, include the following factors:

$$Q_1^+ = (x-y)^3, \tag{11}$$

$$Q_1^- = (x-y)(-2xy + 2y + 2x - 2 + x^2 + y^2), \tag{12}$$

$$Q_2^+ = Q_1^- (-2xy - 2y - 2x + 2 + x^2 + y^2)^3, \tag{13}$$

$$Q_2^- = (x-y)^{12}[11x^4 - y^4 - 24x^3y + (32y - 8 + 14y^2)x^2 + (-8y - 4 - 16y^2)x]. \tag{14}$$

Similarly to the W curves, which inform about singularities in parameter space, Q curves provide useful information concerning singularities of the *orbits* when they bifurcate.

Fig. 2 shows the loci $Q_1^+=0$ and $Q_1^-=0$. The reference curve Q_1^0 corresponds to $\lambda=0$. Q_1^- indicates the location of the (degenerate) period-1 fixed points when the bifurcation 1→2 takes place. Inside the parabola $Q_1^-=0$ one has two period-1 points that coincide along the Q_1^+ diagonal like, say, at p_1. By varying a and b along the dashed line in Fig. 1 (i.e., $(-\frac{1}{4},0)\to(3,\sim\frac{8}{10})$), one fixed-point of period 1 moves from $p_1\to p_2$ along y=x diagonal. Simultaneously, the other period-1 fixed-point moves outside the diagonal, from p_1 to p_3 (which corresponds to the crossing point of the curve c_3 and the dashed line shown in Fig. 1) where it bifurcates into two period-2 points p_5 and p_6. At p_3 the period-1 point is unstable and it moves until p_4, where $a=3$ and $b\sim 0.8$. The curve $Q_2^+=x^2+y^2-2(x+y+xy-1)=0$ in Fig. 2 shows the location where period-2 points are born along c_2.

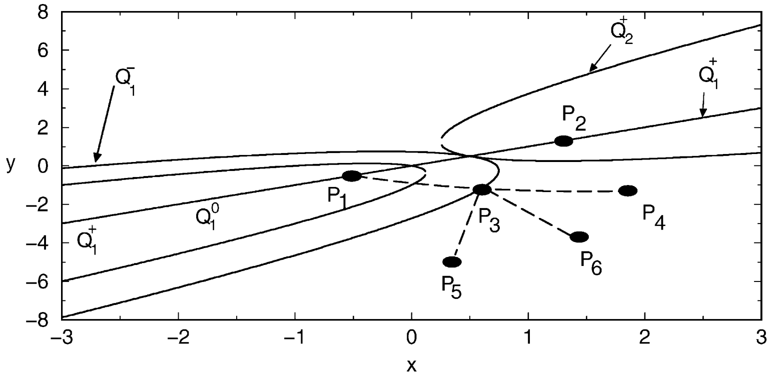

Fig. 2. Bifurcation loci for periods 1 and 2.

In conclusion, bifurcations of two coupled quadratic maps were found analytically to evolve continuously along the $x_n = y_n$ line up to a point where it is possible to move out of this line of symmetry. While the expressions alone for stability boundaries in parameter-space do not give information concerning the position of fixed points in phase-space (and vice-versa), the simultaneous investigation of stability boundaries in both spaces provides relevant informations regarding complex dynamical aspects of novel bifurcations.

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