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Units: remarkable points in dynamical systems

Jason A.C. Gallas [1]

Höchstleistungsrechenzentrum, Forschungszentrum Jülich, D-52425 Jülich, Germany
Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brazil

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Abstract

In number theory, "units" are very special numbers characterized by having their norm equal to unity. So, in the real quadratic field $\mathbb{Z}(\sqrt{3})$ the number $-2+\sqrt{3} \simeq -0.2679491924\ldots$ is a unit because $(-2+\sqrt{3})\,(-2-\sqrt{3}) = 1$. In this paper we determine precisely the numerical values of the coordinates of some *points* defined by multiple intersections of domains of stability in the parameter space of the Hénon map and, in all cases considered for which analytical calculations were feasible, find that such intersection points are invariably defined by units and by simple functions of units. The very special points defined by units are analogous to the familiar multicritical points in phase diagrams. Some simple consequences of the precise dynamics on the ground fields enforced by the equations of motion are discussed.

1. Introduction

The purpose of this paper is to report some analytic results concerning the precise mathematical determination of certain points in the parameter space of simple dynamical systems, namely those points defined by intersections of boundary curves which delimit domains of macroscopically different behaviors. Such intersection points are analogous to the familiar multicritical points which appear commonly in phase diagrams.

The knowledge of the precise location of points of intersections is important because around them one finds several "sectors", or "quadrants", characterized by different physical behaviors. An example of an elementary intersection is illustrated schematically in Fig. 1. This figure shows a two-parameter slice of the parameter space of some generic multi-parameter and multi-variable dynamical system. For definiteness

[1] E-mail: jgallas@if.ufrgs.br

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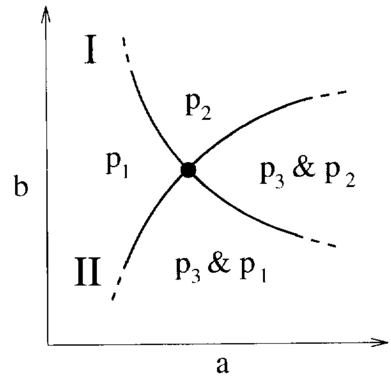

Fig. 1. One point of intersection in parameter space.

and simplicity one may imagine a and b as being the parameters of the Hénon map $(x,y) \mapsto (a - x^2 + by, x)$. Line I in Fig. 1 indicates those parameters for which the system *bifurcates* from a stable motion characterized by some property of interest, say, from a motion characterized by having a periodicity p_1 (on and below I) to a motion characterized by a periodicity p_2 (on and above line I). Line II marks the *birth* (apparently "out of nothing") of stable motion of period p_3, a motion which exists only on and to the right of the line, as indicated by the shading. The shaded region is therefore characterized by the *coexistence* of more than one stable periodic motion (and thus by a much stronger dependence on initial conditions).

As it is easy to recognize from Fig. 1, the intersection of lines I and II defines parameters a and b for which there are *degenerate* stable motions (in a sense to be made precise below). Around the point of intersection one finds four different real and stable physical phenomena:

In the sector p_1	$\leftrightarrow$	existence of motion of period p_1;
In the sector p_2	$\leftrightarrow$	existence of motion of period p_2;
In the sector p_3 & p_1	$\leftrightarrow$	coexistence of motions of periods p_3 & p_1;
In the sector p_3 & p_2	$\leftrightarrow$	coexistence of motions of periods p_3 & p_2.

There are many interesting questions to be asked about the evolution of the dynamics around intersection points. For example, when crossing from left to right the border line I between p_1 and p_2, for parameters lying above the line II we expect basins of attraction to evolve *continuously* in their volume and location. Thus, above line II there is essentially *one* observable "macroscopic" physical change: the change in periodicity $p_1 \to p_2$. In sharp contrast, for parameters located on and below line II there are always two stable attractors such that in addition to a similar change in periodicity, there will be necessarily at least one further macroscopic change in the system: the volume of the basin of attraction for the single attractor existing for parameters located above the line II must display a *discontinuous* change in order to accommodate the basins of the two (instead of only one as previously) stable attractors that coexist on and below the line.

A third macroscopic change that could occur simultaneously with the process of "bifurcation of the *basin* of attraction" happening along the line II is a qualitative change of the basin *boundary*, i.e. of the curve [more precisely, the F-set (see Ref. [1])] delimiting the two new coexisting basins: basin boundaries might be either smooth or "fractal". A discussion of this and additional interesting phenomena that may occur in generic dynamical systems is given in the books by Ott [2], by Nusse and Yorke [3], and in Refs. [4,5]. There is already a fairly good description of possible mechanisms responsible for changes and for the death of chaotic attractors. But a systematic investigation of the precise phenomena occurring during the *limiting processes* involved in the transitions between periodic and aperiodic behaviors seems to be still missing.

In this paper we identify some interesting regions and points in parameter space of the Hénon map,

$$\begin{aligned} x_{t+1} = f(x_t, y_t, a, b) &= a - x_t^2 + by_t, \\ y_{t+1} = g(x_t, y_t, a, b) &= x_t. \end{aligned} \tag{1}$$

For this dynamical system it is relatively easy to progress analytically towards obtaining answers to a number of quite difficult questions concerning the precise dynamics involved in the limiting process involved in transitions between periodic and aperiodic behaviors. The analytical work is done following procedures described elsewhere [1,6], in particular in Section 3 of Ref. [6].

A first example of what we mean by precise analytical results for the dynamics is presented in Fig. 2. This figure shows the first four points of intersection of domains of lowest possible periodicities, determined as described in the sequence of the paper.

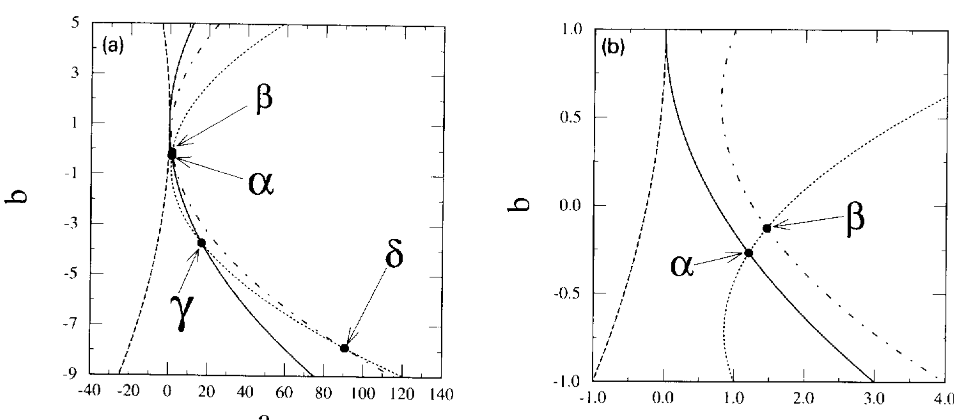

Fig. 2. Four points of multiple intersections along the lines defined by Eq. (13) for the Hénon map. Dashed line: $W_1^+ = 0$; Solid line: $W_1^- \equiv W_2^+ = 0$; Dotted line: $W_3^+ = 0$ and dash-dotted line: $W_2^- = 0$. The dotted line, $W_3^+ = 0$, indicates birth of period-3 orbits, which exist on and to the right of the line. The figure on the right is a magnification of a portion of the view on the left. Stable motions of periods 1, 2 and 3 coexist at the point α while stable motions of periods 2, 4 and 3 coexist at β. Precise numbers defining these points are given in Eqs. (2)–(5).

Points α and β are very close on the scale of the figure on the left. These two points are shown with more detail in the magnification on the right.

The basic result reported in this paper is *the precise number-theoretic identification of the particular parameter values (numbers) and respective degeneracies of the intersections described above and shown in Figs. 1 and 2.* For example, the intersection points α, β, γ and δ in Fig. 2, common to the boundaries of the regions of smallest period, are defined precisely by the algebraic numbers

$$\begin{aligned} a_\alpha &= \tfrac{9}{2}(2-\sqrt{3}), \\ b_\alpha &= -2+\sqrt{3} = 1-\sqrt{6}\sqrt{2-\sqrt{3}} = \tfrac{1}{7}[-5+\sqrt{6}\sqrt{38-21\sqrt{3}}], \end{aligned} \tag{2}$$

$$\begin{aligned} a_\beta &= \tfrac{23}{2}(4-\sqrt{15}), \\ b_\beta &= -4+\sqrt{15} = \tfrac{1}{5}[3-\sqrt{2}\sqrt{452-115\sqrt{15}}] \\ &= \tfrac{1}{7}[-5+\sqrt{2}\sqrt{632-161\sqrt{15}}], \end{aligned} \tag{3}$$

$$\begin{aligned} a_\gamma &= \tfrac{9}{2}(2+\sqrt{3}), \\ b_\gamma &= -2-\sqrt{3} = 1-\sqrt{6}\sqrt{2+\sqrt{3}} = \tfrac{1}{7}[-5-\sqrt{6}\sqrt{38+21\sqrt{3}}], \end{aligned} \tag{4}$$

$$\begin{aligned} a_\delta &= \tfrac{23}{2}(4+\sqrt{15}), \\ b_\delta &= -4-\sqrt{15} = \tfrac{1}{5}[3-\sqrt{2}\sqrt{452+115\sqrt{15}}] \\ &= \tfrac{1}{7}[-5-\sqrt{2}\sqrt{632+161\sqrt{15}}], \end{aligned} \tag{5}$$

where we always mean taking the positive branch of the square-root function. The b values above are defined in $\mathbb{Z}(\sqrt{\Delta})$ with $\Delta = 3$ or 15, i.e. b values are numbers of the generic form

$$u + v\sqrt{\Delta}, \tag{6}$$

with u and v belonging to $\mathbb{Z} \equiv \{0, \pm 1, \pm 2, \pm 3, \dots\}$. In contrast, a values require $\mathbb{Q}(\sqrt{\Delta})$, i.e. have the same form as above but involve rational u and v. The determination of such parameters reduces essentially to the computation of zeros of polynomials, more precisely, to the determination of the proper ground fields where polynomial factorizations occur. Notice that although algebraic parameters like those in Eqs. (2)–(5) may be represented symbolically with precision, it is actually impossible to compute them precisely since they involve infinite sequences of digits. There is an intrinsic undecidability as to where the intersections are located physically in practice.

In all cases where factorization was feasible in practice we find the remarkable result that *the numerical value of the parameters defined by intersections are either* units [7–10] *or simple functions of units in characteristic ground fields.* We believe this result to represent a generic property of dynamical systems. This analytical finding provides a new insight into the exact number-theoretic structure and properties of multicritical points in physical models and allow one to explore important aspects of dynamical systems which are not easily accessible to numerical investigations restricted by finite-precision arithmetics.

The *existence* of units is a well-known fact in number theory [7–10] although the *explicit construction* of units belonging to a given ground field is far from being a trivial problem [9]. In the words of Cohn [10]: "*...the problem of finding units is in general extremely important and also extremely difficult.*" Although there are already a few very interesting books about applications of number theory in physics (for example, Refs. [11–14] and references therein) we are not aware of any previous observation of units occurring in physics. In fact, we are also not aware of any previous application of units outside the abstract domain of number theory.

We conclude this section calling attention to Refs. [15–24] where one finds a number of interesting results concerning iterated maps and number theory as well as additional references.

2. Numerical experiment: lattices and their accumulations

The purpose of this Section is to report a simple numerical experiment performed in the parameter space of the Hénon map. This experiment motivated in fact the derivation of the analytical results given in the continuation of the paper. The Hénon map was chosen because it allows derivation of analytical results with relative ease. The use of this map is not essential since similar results were also obtained for other discrete-time dynamical systems.

The numerical experiment consists of identifying in parameter space the sequence of boundary curves where period-doubling phenomena occur. More specifically, we search for the sequences of boundaries curves resulting from the infinite period-doubling cascades corresponding to stable periodic motions for the two lowest possible periodicities, namely the cascades 1×2^n and 3×2^j, with $n, j = 0, 1, 2, 3, \ldots$. The parameter regions where these two cascades coexist in phase space are shown in Fig. 3. This figure is an isoperiodic diagram obtained as described in Ref. [25]. The cascade $1 \to 2 \to 4 \to \ldots$ is indicated by the colors green $\to$ dark-blue $\to$ red $\to$ $\cdots$ while the cascade $3 \to 6 \to 12 \to \cdots$ is indicated by the sequence of colors: light-blue $\to$ yellow $\to$ green $\to$ $\cdots$. The white region represents "chaos", i.e. parameters for which the period is higher than 32 or for which it was not possible to detect numerically any periodicity. Black represents the basin of attraction of the stable attractor at infinite distance from the origin $(x_0, y_0) = (0, 0)$.

In Fig. 3 it is important to realize that the parabolic arc corresponding to the 3×2^j cascade was painted "over" the cascade corresponding to the 1×2^n doublings. In other words, both cascades "overlap", meaning that for given parameters in the region of overlap one may observe motions belonging to either one of the coexisting cascades, depending only on the initial conditions. Further, we observe that it is also possible to find motions with stable periodicities other than 1×2^n and 3×2^j in certain sub-regions of the region of overlap. But this fact is not relevant for our present purposes.

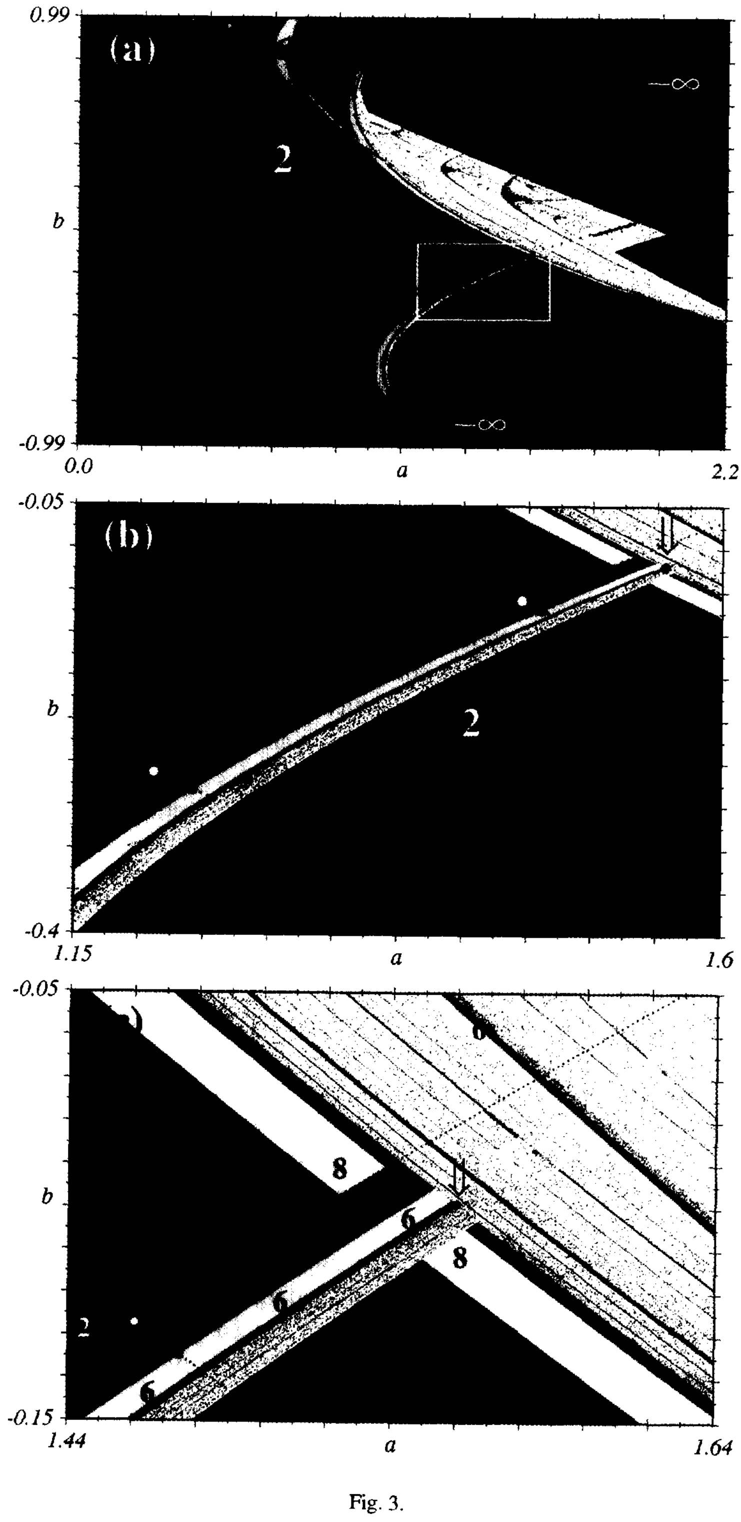

Fig. 3.

To enhance contrast between adjacent parameter regions we sometimes used the same colors to represent regions of different periodicities. This may be more easily recognized in Fig. 3c where yellow is used to represent simultaneously regions of parameters leading to motions of periods 6 or 8. The 3×2^j cascade was painted over the 1×2^n cascade only up to b values where the 3×2^j cascade meets the parameters beyond which the cascade 1×2^n accumulates into aperiodic motions. Further, as a guide to the eye, some of the varieties defined in Eq. (13) (defined below) were superimposed as dotted lines in Fig. 3. The characteristic stripes seen in the figure to correspond to the 3×2^j cascade continue to exist along the dotted line seen on the upper-right sector of Fig. 3c. By using different initial conditions one could easily paint them on the figure, instead of the behaviors actually shown.

From Fig. 3 one recognizes the existence of a two-dimensional nonlinear mesh of points which accumulate on the point $\mathcal{A}_{1,3}$ indicated by the arrow in Figs. 3b and 3c. The first four points of this doubly-infinite mesh are indicated by large dots in Fig. 3b, two white dots [along the line of birth of period-3 motion], and two black dots [along the $3 \rightarrow 6$ bifurcation line]. The two white dots are the points α and β shown earlier in Fig. 2 and defined by Eqs. (2) and (3). The two additional black dots are the points α' and β' defined in Table 1 below (Section 6).

By properly choosing initial conditions one may generate figures similar to Fig. 3, showing nonlinear lattices of periodicities $k_1 \times 2^n$ and $k_2 \times 2^j$ with accumulation points $\mathcal{A}_{k_1,k_2}$. One may also expect to find accumulation points generated as the common limit of more than two coexisting lattices.

A number of questions arise naturally from the realization of the existence in parameter space of dynamical systems of a great profusion of accumulation points [along with the generalized "comet-like" nonlinear lattices leading to them]. What universal behaviors and scalings should be expected when one dares to go beyond the familiar one-parameter situation and venture considering *tuning more than one parameter simultaneously* [e.g. move along one of the lines I or II in Fig. 1] and contemplate the individual cascades of changes induced by the accumulation of more and more points of intersection? Which are the "practical limits" that constrain experimental detectability of parameter intervals that very quickly have essentially measure zero for most experimental situations? How to derive appropriate "uncertainty relations" telling precisely where our ability to discriminate between different physical behaviors ends as a function of the experimental resolution? How to know precisely (from a theoretical point of view at least) when we arrive at some pre-assigned distance from a *double* accumulation

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Fig. 3. Intersection points along boundary curves delimiting regions of different periodicities of the Hénon map. Integers indicate the periodicities of the larger regions. The sequence of intersection points forms a two-dimensional lattice (see discussion in Section 2). The arrow indicates the accumulation point $\mathcal{A}_{1,3}$ of the two-dimensional lattice formed by the two cascades of lowest possible periodicities: 1×2^n and 3×2^j, a "Mächtigkeit 2" accumulation point. Parameter values defining the location of the four large dots in (b) are given in Table 1 below. Details in Fig. 3a are discussed in Refs. [25,26].

point [such as the one indicated by the arrow in Figs. 3b and 3c] or at more general accumulations? Which are the precise number-theoretic conditions determining accumulation points formed by the coexistence of an arbitrary number of lattices (cascades)? Which number-theoretic mechanisms and ground fields determine the character of basin boundaries, i.e. their smooth or "fractal" nature?

In the remainder of this paper we dwell upon two main problems: (i) to set up a framework for the analytical determination of generic algebraic varieties similar to those seen in Fig. 3 which define border lines between regions characterized by different physical properties, and (ii) the explicit determination of the number-theoretic nature of the first few relevant intersection points for one specific example: the Hénon map. The intersections seen in Fig. 3 arise from very specific *reciprocity relations* between the physical parameters (numbers) underlying and ruling the dynamics as we now demonstrate. These reciprocities are only possible when all parameters involved belong to the same ground field and are connected by certain relatively simple relations of linear dependence in the ground field.

3. How to obtain the boundaries between domains of stability

We now discuss a general procedure to obtain analytical expressions for the varieties defining boundaries between domains characterized by different stable motions in parameter space of generic two-dimensional algebraic dynamical systems. This procedure is essentially a simple generalization of the approach discussed in Section 3 of Ref. [6], and has its origins in works by Euler (1707–1783), and with much more amplitude and accuracy by Bézout (1730–1783), [in his *Théorie générale des équations algébriques,* Paris, 1779] and Sylvester (1814–1897) among others. Euler mentions [in chapter XIX "de intersectione curvarum" of *Introductio in analysin infinitorum,* Lausanne, 1748] that the same method of elimination was already used by Newton (1642–1727). The procedure for obtaining varieties discussed here may be further extended to higher-dimensional systems with no difficulty. The procedure consists of two steps.

First step ("iteration step"): it consists of evaluating the k-fold composition of the equations of motion as well as computing the determinant M_k of the 2×2 matrix $J_k(x,y,a,b) - m \cdot I$, i.e.

$$M_k(x,y,a,b,m) \equiv |J_k(x,y,a,b) - mI|, \tag{7}$$

where $J_k(x,y,a,b)$ is the Jacobian matrix of the composition, I represents the standard 2×2 identity matrix and m is the "multiplier", i.e. the eigenvalue-parameter which controls the stability of the k-periodic solution [6]. Notice that more generally one would need to write m_k instead of m in order to account for the fact that multipliers associated with different orbits, in particular orbits with different periodicities, are independent quantities. For simplicity, we omit one or more additional indices here.

For every generation k one obtains in this way *three* equations, each one of them involving the variables x and y and parameters a and b, equations which may be written formally as

$$\begin{aligned} X_k(x,y,a,b) \equiv x - f_k(x,y,a,b) &= 0, \\ Y_k(x,y,a,b) \equiv y - g_k(x,y,a,b) &= 0, \\ M_k(x,y,a,b,m) &= 0, \end{aligned} \tag{8}$$

where k in f_k and g_k labels the number of compositions to be performed in order to obtain the equation, and $f(x,y,a,b)$ and $g(x,y,a,b)$ define an arbitrary dynamical system. In other words, k represents the periodicity of the solution-set (x,y,a,b) under consideration. For a given period k and parameters a and b, the first two equations, $X_k = 0$ and $Y_k = 0$, define the physical solutions (x,y) while the last equation, $M_k = 0$, through the magnitude of m, defines the relative stability of these solutions. As discussed in Ref. [26], a given solution is stable for all m values in the interval $-1 \leq m \leq 1$.

Second step ("elimination step"): it consists of eliminating variables. For example, we may eliminate y between X_k and M_k and between Y_k and M_k, ending up with two further equations, each one of them involving only the quantities (x,a,b,m). Eliminating x between these two equations we obtain, finally, a single equation involving only the parameters a and b, the multiplier m and several *integer* numerical coefficients resulting from rather complicated combinatorial problems, i.e. an equation that we write as

$$W_k(a,b,m) = 0. \tag{9}$$

The equation $W_k(a,b,m) = 0$ is the important equation in the present context: *it contains the maximum possible information concerning the parameter space and, simultaneously, the stability of the system for every period k.* This equation defines the *ground field* for all k-periodic dynamics in phase space. The above procedure to obtain $W_k(a,b,m)$ is general and applies, *mutatis mutandis*, equally well to multiparameter higher dimensional systems.

Having determined $W_k(a,b,m)$, one may easily delimit intervals of stable k-periodic motions in parameter space by simply plotting the varieties defined by

$$W_k^- \equiv W_k(a,b,m=-1) = 0, \qquad W_k^+ \equiv W_k(a,b,m=+1) = 0. \tag{10}$$

In the current literature it is costumary to work with the absolute value of m, searching numerically for those (a,b) for which $|m| = 1$. It is also costumary to address questions of stability by formulating them as eigenvalue problems. Rather than working with the absolute value of m, we prefer to investigate two cases separately by setting either $m = -1$ or $m = +1$ right at the beginning of the calculations. [Eventually, one may also set $m = 0$ whenever this additional case might be of interest, always after suitable divisions in order to avoid false zeros.] After fixing the value of m we study the corresponding varieties in the $a \times b$ plane. In other words, rather than computing $|m|$ for every (a,b), we fix m suitably (not $|m|$) and compute the corresponding (a,b) varieties. As it will be seen in the next Section, for the lowest periods this procedure may be performed analytically using softwares presently available to do algebraic manipulations with computers. This allows one to obtain important number-theoretic insight about the precise dynamics on the ground field and to pose and investigate new problems.

Note that in our alternative approach, rather than investigating stability by computing eigenvalues in the familiar manner, the stability problem is reformulated in a way to have the eigenvalue embedded as an "additional parameter" (the stability parameter m) at the level of the equation $W_k(a,b,m)$ discussed above. The parameters a and b solving $W_k(a,b,m) = 0$ will be obviously implicit functions of m and, at least formally, may be always written as $a = a(m)$ and $b = b(m)$. These solutions may be used to embed m directly into the original equations of motion, allowing one to work in phase space exclusively with the equations of motion [i.e. without having to consider subsequently eigenvalue problems], looking for specific solutions having their stability fixed *ab initio* by m. Similar treatment is possible when the number of parameters and/or variables is greater than two.

We conclude this Section observing that by eliminating either x or y between $X_k(x,y,a,b)$ and $Y_k(x,y,a,b)$ one obtains polynomial equations defining *all possible solutions $x = x(a,b)$ [and/or $y = y(a,b)$] in phase space* as functions of the parameters. The polynomials defining solutions will be denoted either by $\mathcal{P}_k(x,a,b)$ or $\mathcal{P}_k(y,a,b)$, depending on which variable remains after the elimination. For the Hénon map, the polynomials obtained by eliminating x are identical with those obtained by eliminating y, a trivial simplifying consequence of the equation $y_{t+1} = x_t$.

4. Prime factors define classes of physical trajectories

We now return to the specific example of the Hénon map (Eq. (1)).

Apart from simple constant multiplicative factors, by eliminating y between $Y_k(x,y,a,b)$, one obtains for the Hénon map the following polynomial equations which define all possible solutions $x = x(a,b)$, stable or not, for the first generations $k = 1$, 2, 3 and 4:

$$\begin{aligned}
\mathcal{P}_1 &= a - (1-b)x - x^2,\\
\mathcal{P}_2 &= a - a^2 - 2ab + ab^2 - (1 - 3b + 3b^2 - b^3)x + 2ax^2 - x^4,\\
&= [a - (1-b)x - x^2][1 - a - 2b + b^2 - (1-b)x + x^2],\\
\mathcal{P}_3 &= [a - (1-b)x - x^2]\left(\sum_{j=0}^{6} c_j x^j\right),\\
\mathcal{P}_4 &= [a - (1-b)x - x^2][1 - a - 2b + b^2 - (1-b)x + x^2]\left(\sum_{j=0}^{12} d_j x^j\right).
\end{aligned} \tag{11}$$

Explicit expressions for the coefficients c_j and d_j are given in the Appendix. In general, as the periodicity k increases, all polynomials $\mathcal{P}_k(x,a,b)$ consist of products of a certain number of prime factors $p_{k,j}(x,a,b)$ appearing with multiplicities $\mu_{k,j}$:

$$\mathcal{P}_k = p_{k,1}^{\mu_{k,1}} p_{k,2}^{\mu_{k,2}} p_{k,3}^{\mu_{k,3}} p_{k,4}^{\mu_{k,4}} \cdots p_{k,\ell}^{\mu_{k,\ell}}, \tag{12}$$

where all dependencies in (x,a,b) are omitted for simplicity. Clearly, the new prime factors contribute additional ground fields and zeros of the polynomials, zeros that for suitable "in-phase" combinations (see Section 7b, below) build new possible physical trajectories as iterations proceed. Notice that the polynomial $\mathcal{P}_1(x,a,b)$ divides all subsequent polynomials $\mathcal{P}_k(x,a,b)$, i.e. the prime factor $\mathcal{P}_1(x,a,b)$ defining the fixed points plays the same role played by the number 1 in the factorization of integer numbers. Notice further that $\mathcal{P}_k$ is a divisor of $\mathcal{P}_{2k}$ (and necessarily of all other $\mathcal{P}_j$ belonging to the infinite $k \times 2^n$ cascade).

All possible trajectories with periods less than or equal to 4 [stable or not, real or not] involve points which must be necessarily zeros of the prime polynomial factors above. Thus, all possible phase space dynamics will be ruled necessarily by the ground field enforced by the parameters appearing in these factors, here a and b in Eqs. (11). Further, one recognizes the mathematical property which underlies the two possible regimes, periodic or not, for phase space dynamics: the only way for a motion to be periodic or to become periodic [after a *finite* number of iterates] is if it starts from initial conditions belonging to the ground field enforced by the parameters. For initial conditions not in the same field the motion must remain transient forever, i.e. will be always just *asymptotic* to one of the possible motions corresponding to exact zeros. From an experimental point of view this rather strict reality will be frequently subtle, if not impossible, to detect since observations limited by finite resolutions make it hard to distinguish accurately between "moving toward the final attractor" and "actually being on the final attractor". While the difficulty in distinguishing between these possibilities is certainly very familiar, the new thing that emerges from the discussion above is that the physical ability to discriminate between possible final states, "predict the future", seems to be fundamentally associated with (and limited by) the ability of ascertaining in practice numerical values of parameters and initial conditions with precision right at the beginning. In the absence of perturbations, there is absolutely no mechanism allowing changes of the underlying ground field.

An important quantity discriminating physical motions in phase space is the length of time, the "transient", needed to go from a given initial condition to the final attractor. Transient times will vary between zero [for dynamics starting from initial conditions given exactly by appropriate combinations of zeros of the polynomials] and infinite [for example, for all initial conditions not in the field of the coefficients]. Finite transient times of any arbitrary length occur for initial conditions defined over the ground field and which are pre-images of the zeros of the polynomials. The actual length of the transient is defined in a simple way by the "embedding-depth" of the pre-images in the ground field, this depth being clearly related to the actual "distance" in number of iterates needed to go from the initial condition to the final orbit or, at least, very close to it.

From the equations above one recognizes that the family of polynomials $\mathcal{P}_k$ are invariably defined by products of certain prime factors p_k, in perfect analogy with the familiar factorization of integer numbers in terms of prime numbers. Thus, to study domains of k-periodic motion means studying the algebraic varieties defined by each of the p_k factors. In particular, to study a particular "route to chaos" means looking for the possible ways of connecting degenerate varieties [i.e. varieties which are common to two consecutive p_k factors] as the iteration proceeds.

From the equations above one may also recognize that it is possible to regard the Hénon map as being one-dimensional rather than a two-dimensional dynamical system. This statement is also valid for other dynamical systems of dimension two or higher. *The effect of the extra dimension is simply to produce the specific coefficients appearing in an "one-dimensional equivalent" system obtained by eliminating all but one variable.* Analogously, higher-dimensional systems may be interpreted simply as allowing more flexible and complicated interconnections between all parameters appearing in the coefficients of the one-dimensional equivalent map. The specific relations among parameters are obviously of fundamental importance in determining the prime factors composing the several $\mathcal{P}_k$ and their respective ground fields.

Instead of embarking now on a generic investigation of the parameter-dependence of the zeros of the prime factors introduced in this Section, we proceed, first, to determine "important" parameter values and their corresponding ground fields and only then, second, to investigate the dynamics generated by the zeros corresponding to these parameters. Here, particularly important parameters are those common to as many regions of different physical behaviors as possible. It is clear that the prime factors discussed in this Section are generic characteristics of all dynamical systems defined by polynomial equations of motion, not a particularity of the Hénon map.

5. Prime factors define boundaries between domains of stability

We now apply the procedure described in Section 3 to the equations derived in Section 4, obtaining in this way all important varieties delimiting in parameter space stability regions of periods 1, 2 and 3. To reduce the size of the formulas, we present here only those irreducible prime factors that survive divisions by factors of lower orders. These are the only factors explicitly needed for our present purposes. The factors are:

$$\begin{aligned}
W_1^+ &= -4a - 1 + 2b - b^2, && \text{(Birth of period 1)}\\
W_1^- &= -4a + 3 - 6b + 3b^2, && \text{(Bifurcation } 1 \to 2)\\
W_2^- &= -4a + 5 - 6b + 5b^2, && \text{(Bifurcation } 2 \to 4)\\
W_3^+ &= -4a + 7 + 10b + 7b^2, && \text{(Birth of period 3)}\\
W_3^- &= \quad 64a^3 - 32(4 + b + 4b^2)a^2 \\
&\quad +(72 - 216b - 252b^2 - 216b^3 + 72b^4)a \\
&\quad -81 - 54b - 18b^2 + 90b^3 - 18b^4 - 54b^5 - 81b^6 \\
& && \text{(Bifurcation } 3 \to 6)
\end{aligned} \tag{13}$$

All expressions above are exact. The expression for W_1^- coincides with that for W_2^+. Notice the conspicuous fact that all polynomials in the parameter b [i.e. all coefficients of the equations defining a] are *reciprocal* polynomials, i.e. polynomials $p(b) = 0$ of degree n which satisfy

$$b^n p(1/b) = \pm p(b). \tag{14}$$

Table 1
Approximate locations of the four intersections marked by large dots in Fig. 3b. The exact value of the parameter indicated by the asterisk is defined by b_1 in Eq. (17). Points α and β are defined in Eqs. (2) and (3), respectively. α' is the point represented by the black dot located near α

Point	Equation	Approximate location
α	$R[W_1^-, W_3^+, b] = 0$	$a \simeq 1.20577136594005$
	$R[W_1^-, W_3^+, a] = 0$	$b \simeq -0.267949192431122$
α'	$R[W_1^-, W_3^-, b] = 0$	$a \simeq 1.232588541359$
	$R[W_1^-, W_3^-, a] = 0$	$b \simeq -0.281971680061195$*
β	$R[W_2^-, W_3^+, b] = 0$	$a \simeq 1.4606915186147$
	$R[W_2^-, W_3^+, a] = 0$	$b \simeq -0.127016653792583$
β'	$R[W_2^-, W_3^-, b] = 0$	$a \simeq 1.47441076259914$
	$R[W_2^-, W_3^-, a] = 0$	$b \simeq -0.134526112591863$

In general, as the periodicity k increases, all varieties defined by $W_k = 0$ originate from certain prime factors $w_{k,j}$ appearing with multiplicities $\nu_{k,j}$ in W_k:

$$W_k = w_{k,1}^{\nu_{k,1}} w_{k,2}^{\nu_{k,2}} w_{k,3}^{\nu_{k,3}} w_{k,4}^{\nu_{k,4}} \cdots w_{k,\ell}^{\nu_{k,\ell}}, \qquad (15)$$

where the multiplicities $\nu_{k,j}$ of each factor might increase quickly when the period k increases. In fact, rather than W_k, what is given in Eq. (13) are simply the appropriate w_k required for the present applications. Doubling bifurcations correspond to parameters defined by zeros of "degenerate" prime factors w, namely by factors w appearing simultaneously in both W_k^- and W_{2k}^+. Parameter values defined by zeros of non-degenerate prime factors produce invariably new regions in parameter space, regions in which independent $k \times 2^n$ cascades are typically born. Here "independent cascades" means clusters of bifurcations not sharing a common boundary with any previously existing cascade.

In Refs. [25,26] we had the opportunity to discuss the great regularity with which shrimp-like regions [characterized in phase space by having isoperiodic attractors] appear in parameter space. We use the word "shrimp" [25,26] to denote the union of all adjacent cells in parameter space corresponding to the interval between creation and annihilation of a full $k \times 2^n$ cascade in phase space. In this case, *odd values* of k imply necessarily the birth of a new shrimp independent of all previous shrimps while *even values* might imply either the birth of a whole new shrimp or just a mere bifurcation between adjacent cells within a pre-existing shrimp.

6. The precise number-theoretic nature of multiple intersections

In the previous Section we have determined all relevant prime factors defining the boundaries of stability regions in parameter space for motions of the three lowest possible periodicities. We now proceed to determine the multiple intersection of these regions. To this end we start by defining an operator to represent the operation of computing the *resultant* [27] between two polynomials with respect to an indeterminate common to them.

Given a pair of arbitrary algebraic functions $f = f(a,b)$ and $g = g(a,b)$ depending on two indeterminates a and b we will use the symbol $\mathcal{R}[f,g,v]$ to represent the result of the evaluation of their resultant, where v represents either one of the indeterminates, $v = a$ or $v = b$. Explicit algorithms for the computation of resultants are discussed in standard texts of algebra, for example in Ref. [27].

It is a simple exercise to show that the resultants of interest are given by the following expressions (apart from "normalization", i.e. overall multiplicative constants):

$$
\begin{aligned}
\mathcal{R}[W_1^-, W_3^+, a] &= 1 + 4b + b^2 = (b - b_\alpha)(b - b_\gamma),\\
\mathcal{R}[W_1^-, W_3^+, b] &= 81 - 72a + 4a^2 = 4(a - a_\alpha)(a - a_\gamma),\\
\mathcal{R}[W_2^-, W_3^+, a] &= 1 + 8b + b^2 = (b - b_\beta)(b - b_\delta),\\
\mathcal{R}[W_2^-, W_3^+, b] &= 529 - 368a + 4a^2 = 4(a - a_\beta)(a - a_\delta),\\
\mathcal{R}[W_1^-, W_3^-, a] &= [1 + b + b^2](1 + 2b - 5b^2 + 2b^3 + b^4),\\
\mathcal{R}[W_1^-, W_3^-, b] &= [81 - 36a + 16a^2]\\
&\quad \times (81 + 1080a + 3888a^2 - 4224a^3 + 256a^4),\\
\mathcal{R}[W_2^-, W_3^-, a] &= 33 + 226b - 114b^2 + 200b^3 - 114b^4 + 226b^5 + 33b^6,\\
\mathcal{R}[W_2^-, W_3^-, b] &= 61168041 - 368275248a + 1089069012a^2,\\
&\quad -878940000a^3 + 334497472a^4\\
&\quad -94827008a^5 + 1115136a^6,
\end{aligned}
\tag{16}
$$

where the precise numerical values of a_α, a_β, a_γ, a_δ, b_α, b_β, b_γ and b_δ are those already given in Eqs. (2)–(5). The approximate numerical location of the four intersections indicated by large dots in Fig. 3b are given in Table 1.

All points of multiple intersections are defined by the zeros of the resultants above and similar ones. These same zeros define also the specific ground fields enforcing phase space dynamics.

Notice the existence of simple relations among the apparently independent parameters for which multiple intersections occur:

$$2a_\alpha + 9b_\alpha = 0, \qquad 2a_\beta + 23b_\beta = 0.$$

Identical relations are satisfied by their conjugate values a_γ and a_δ:

$$2a_\gamma + 9b_\gamma = 0, \qquad 2a_\delta + 23b_\delta = 0.$$

The prime factors $1+b+b^2$ and $81-36a+16a^2$, which appear inside square brackets, produce only complex parameters and orbits and therefore will not be considered here.

The next prime factor of special interest for the present purposes is given by $1+2b-5b^2+2b^3+b^4$, which factors into the following parameters:

$$b_1 = -\tfrac{1}{2} - \sqrt{2} + \tfrac{1}{2}\sqrt{5+4\sqrt{2}} \simeq -0.2819716801\ldots,$$

$$b_2 = -\tfrac{1}{2} - \sqrt{2} - \tfrac{1}{2}\sqrt{5+4\sqrt{2}} \simeq -3.5464554447\ldots,$$

$$b_3 = -\tfrac{1}{2} + \sqrt{2} + \tfrac{1}{2}\sqrt{5 - 4\sqrt{2}} \simeq 0.91421 + i0.4052,$$
$$b_4 = -\tfrac{1}{2} + \sqrt{2} - \tfrac{1}{2}\sqrt{5 - 4\sqrt{2}} \simeq 0.91421 - i0.4052. \qquad (17)$$

Notice that $b_1 b_2 = b_3 b_4 = 1$, i.e. being zeros of an irreducible reciprocal equation, all parameters above are automatically units.

The zeros of $81 + 1080a + 3088a^2 - 4224a^3 + 256a^4$ lead to similar expressions for the parameters.

We conclude the determination of the multiple intersections by observing that using the substitution $B \equiv b + 1/b$ [or, eventually, $\tilde{B} \equiv b - 1/b$] and the fact that in general (with the sum running over all $j \leq n/2$ which are non-negative integers)

$$b^n + \frac{1}{b^n} = \sum_j (-1)^j \frac{n}{n-j} \binom{n-j}{j} B^{n-2j}, \qquad (18)$$

all zeros of the sextic $33 + 226b - 114b^2 + 200b^3 - 114b^4 + 226b^5 + 33b^6$ might be analytically expressed in terms of the zeros of a cubic. Since this sextic is an irreducible reciprocal polynomial, all its zeros are automatically units.

As noticed above, for the first few prime factors it is quite easy to show that both parameters a and b are not independent from each other, being connected through simple rational factors, while the transformation of a polynomial in a into the corresponding one in b in this case is a simple exercise. An interesting open problem is the determination of specific transformations that would allow one to pass generically, for arbitrary periodicities, from polynomials in one parameter (usually reciprocal polynomials) into the corresponding polynomial in another parameter (usually "cousin" of a reciprocal polynomial), i.e. to determine the nature of the dependency between parameters underlying the multiple intersections. It seems that multiple intersections imply the existence of subtle but not too complicated interconnections, linear dependencies, between parameters. Parameters defined by multiple intersections seem to be always expressible by specific linear combinations on the characteristic ground field.

7. About the actual computability of physical trajectories

So far we have worked *exclusively in parameter space*, a characteristic of the approach introduced in Section 3. In the present Section we wish to fix parameter values and consider phase space dynamics. Our main purpose will be to call attention to the need of using just *all possible combinations of the zeros of the prime factors* defining trajectories to probe properties of dynamical systems and of their corresponding ground field. We also discuss some intrinsic properties of algebraic numbers that will necessarily limit the accuracy of experiments, both in the laboratory and with computers. We will consider one single very specific point in parameter space: the first multiple intersection occurring at

$$(a_\alpha, b_\alpha) = ((9/2)[2 - \sqrt{3}], -2 + \sqrt{3}). \qquad (19)$$

The discussion is similar for other analogous points in parameter space.

7.1. Determination of the three stable attractors at finite distances

The fixed points (period-1 orbits) of the equations of motion for (a_α, b_α) are given by

$$x_\pm = y_\pm = -3/2 + (\sqrt{3}/2) \pm \sqrt{6}\sqrt{2-\sqrt{3}}. \tag{20}$$

Observing that $48 - 24\sqrt{3} \equiv (6 - 2\sqrt{3})^2$ is a perfect square in $\mathbb{Z}(\sqrt{3})$ one may also write the fixed points in a simpler form:

$$x_+ = y_+ = (1/2)(3-\sqrt{3}), \qquad x_- = y_- = -(3/2)(3-\sqrt{3}). \tag{21}$$

The stable fixed point is located at $s \equiv (x_+, y_+)$ while the unstable fixed point lies at $u \equiv (x_-, y_-)$ where $x_+ = y_+ \simeq 0.6339746\cdots$ and $x_- = y_- \simeq -1.9019238\cdots$.

From the second quadratic factor in $\mathcal{P}_2$ in Eq. (11) one obtains the two zeros which define the period-2 orbit. For (a_α, b_α) this *different* quadratic factor has a double zero identical to x_+:

$$1 - a_\alpha - 2b_\alpha + b_\alpha^2 - (1-b_\alpha)x + x^2 = \{x - \tfrac{1}{2}(3-\sqrt{3})\}^2. \tag{22}$$

The last stable orbit of interest that remains to be computed is the period-3 orbit. As discussed in Appendix A, the sextic prime factor in $\mathcal{P}_3$ degenerates into a cubic equation along the variety $W_3^+ = 0$. The specific cubic for (a_α, b_α) is

$$9 - 5\sqrt{3} - 18(2-\sqrt{3})x - 2(3-\sqrt{3})x^2 + 4x^3 = 0. \tag{23}$$

Exact analytical expressions for the zeros may be obtained from the known formulas but are too long to write down explicitly here and we give just numerical approximations to them, which already indicate their proper "in-phase" combinations leading to the period-3 orbit (x on top, y bottom):

$$\cdots \to \begin{pmatrix} 1.433682753 \\ 0.0682557999 \end{pmatrix} \to \begin{pmatrix} -0.867963956 \\ 1.433682753 \end{pmatrix} \to \begin{pmatrix} 0.0682557999 \\ -0.867963956 \end{pmatrix} \to \cdots$$

The underlying ground field enforced in phase space by the parameters in this case is clearly $\mathbb{Q}(\sqrt{3})$.

7.2. In-phase and out-of-phase dynamics in a field

The prime factors obtained by eliminating x or eliminating y between the equations of motion will be always identical for dynamical systems involving an equation of motion of the type $y_{t+1} = x_t$ in their definition. This is the case of the Hénon map. Such degeneracy between the polynomials defining all possible x and y values is welcome because it simplifies the analysis enormously. For example, instead of having to consider the following four combinations of zeros as possible candidates that might lead to fixed points:

$$s \equiv \begin{pmatrix} x_+ \\ y_+ \end{pmatrix}, \qquad p \equiv \begin{pmatrix} x_+ \\ y_- \end{pmatrix}, \qquad q \equiv \begin{pmatrix} x_- \\ y_+ \end{pmatrix}, \qquad u \equiv \begin{pmatrix} x_- \\ y_- \end{pmatrix}, \tag{24}$$

we only need to consider the pair of combinations $s \equiv (x_+, y_+)$ and $u \equiv (x_-, y_-)$. But for generic dynamical systems, after determining all zeros of both polynomials in x and in y, one still needs to investigate which are the proper *in-phase combinations* of these zeros that when used as initial conditions will produce fixed points. In other words, *in order to solve completely the problem concerning existence of periodic orbits, stable or not, one still needs to investigate the dynamics for the lattice of all possible combinations of the zeros when used as initial conditions.* Obviously, the reduction of the number of initial conditions to just all possible combinations of the zeros of the prime factors corresponding to each individual variable is a generic property of all the "equivalent one-dimensional systems" discussed in Section 4, not a particularity of the degenerate prime factors being presently considered.

Fig. 4 shows basins of attraction and corresponding attractors for the parameters (a_α, b_α). In this figure there are two rectangular lattices of points formed by combinations of zeros of prime factors: four larger points indicated by the letters s, u, p and q and nine smaller points, three of which are indicated by the numbers 1, 2 and 3. The four larger show initial conditions as defined in Eq. (24), s representing the stable and u the unstable fixed points. The nine smaller points form the rectangle of initial conditions defined by combinations of the zeros of the cubic in Eq. (23) . The cubic defines the stable orbit of period 3: $\cdots \to 1 \to 2 \to 3 \to 1 \cdots$, where the numbers correspond to the points labeled in the figure. These rectangles of points display typical characteristics that are observed for other parameter values: (i) very few initial conditions need to be considered in establishing appropriate in-phase combinations of zeros leading to stable dynamics, and (ii) some points forming the lattices will lie in the basin of attractors defined by points of the *same* lattice while others will lie in basins belonging to attractors defined by points of a different lattice.

Since all points belonging to each generic lattice are defined on a common ground field, from a number-theoretic point of view one is naturally induced to ask several questions about the exact dynamics, the most basic ones being: (i) Which are the precise conditions characterizing those very specific combinations of zeros that when iterated move away from the attractor defined on the same lattice? In other words, which properties distinguish points like q from points u, p and s? (ii) For those points lying inside the basin of attractors belonging to the same lattice, how many iterates are they "away" from the stable attractor? More precisely, when starting iterations from points like p, how many iterates are necessary in order to enter a circle of an arbitrary radius ϵ centered in s? Under which conditions may this circle have radius zero, i.e. may we end up precisely on the stable attractor after just a finite number of iterates? (iii) What properties are present in the sequences of algebraic numbers generated by the dynamics while moving either towards or away from attractors belonging to a common lattice?

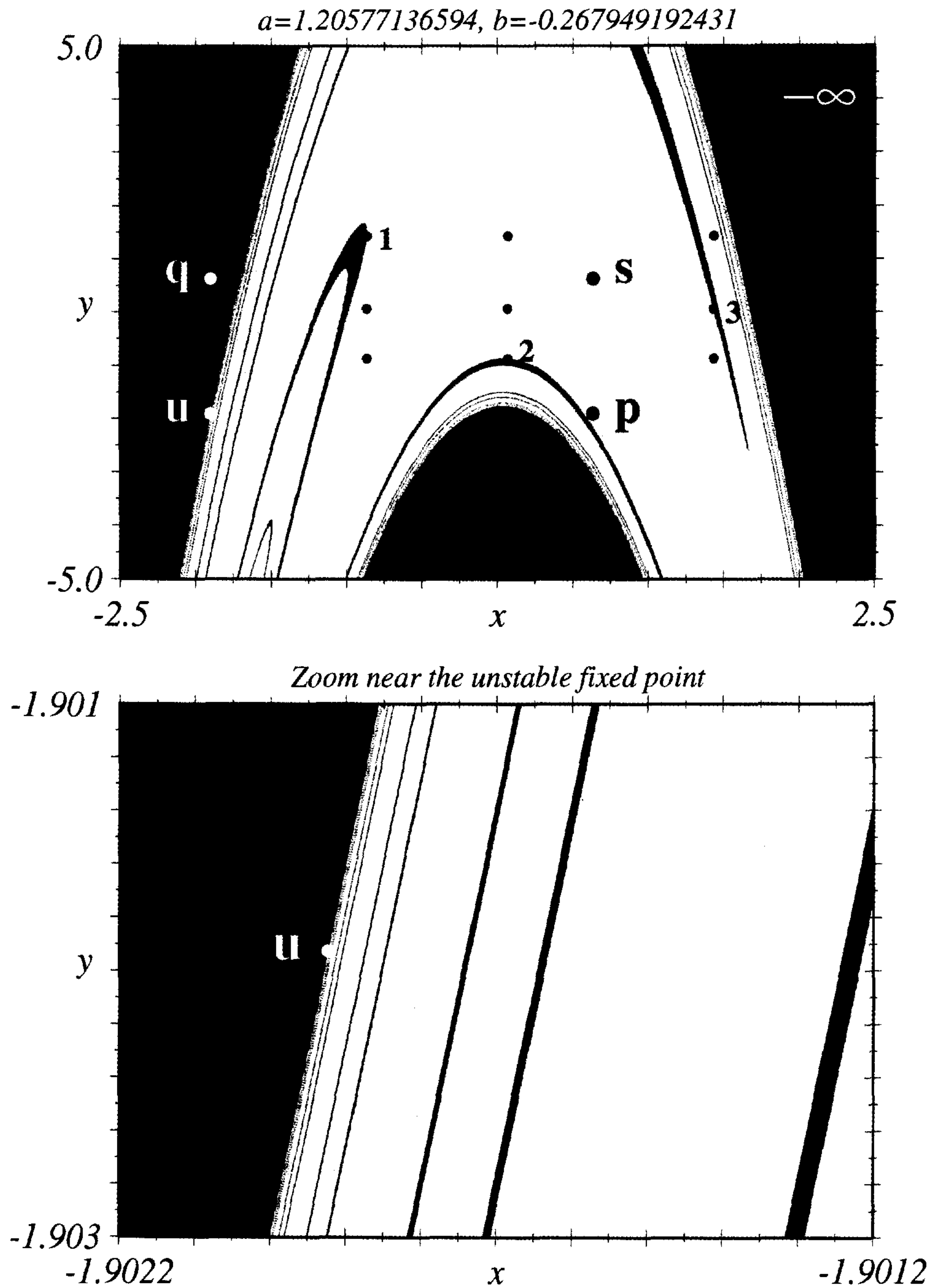
a=1.20577136594, b=-0.267949192431
5.0
-5.0
-2.5
2.5
x
y
1
2
3
q
s
u
p
—∞
Zoom near the unstable fixed point
-1.901
-1.903
-1.9022
-1.9012
x
y
u

Fig. 4.

Table 2
The first few exact values of x_t for (a_α, b_α) when started from the initial combination of zeros denoted by p in Eq. (24) and in Fig. 4. The actual points of the orbit are (x_{t+1}, y_{t+1}), where $y_{t+1} \equiv x_t$, $t = 0, 1, 2, \ldots$ and $y_0 = (-9 + 3\sqrt{3})/2$. All x_t are given in the form $x_t = (U + L\sqrt{3})/2$, where U and L refer respectively to the "upper" and "lower" integer "coordinates" shown in the table. For example, $x_1 = (39 - 21\sqrt{3})/2$. The factorizations indicated were obtained with the commercial software Mathematica. Mathematica did not produce any factorization for the last numbers (although one is obviously not a prime) after running during 100 hours on a SUN Sparc 10 workstation, when the run was manually interrupted. The last column shows finite-precision approximations of the exact numbers

t	Exact values of x_t	Approximation
0	3	0.633975
	-1	
1	3×13	1.313467
	-3×7	
2	$-3^2 \times 157$	−0.689296
	5×163	
3	$-3 \times 5 \times 71 \times 1873$	0.3787002
	$3^2 \times 11 \times 11633$	
4	-3×1326336373519	1.247054
	$1103 \times 1949 \times 1068629$	
5	$-3^2 \times 5897 \times 9931 \times 30038846397282763$	−0.4508442
	$3 \times 59 \times 163 \times 316831512884100490553$	
6	$-250668486307929610494856984164329220908603061584373$	0.6683638
	$1447235180472391808067631486860678706965349061684 15$	

(iv) Which is the nature of the set of pre-images leading to points like p and q?

We have investigated these and related questions and will report the results elsewhere. Here, to give an idea of the sort of insight provided by the exact dynamics, we would like to mention briefly two results: first, that the exact dynamics is an efficient generator of quite large prime numbers and, second, that trajectories started from almost all initial conditions will require an infinite amount of time (i.e an infinite number of iterates) in order to reach their respective final attractors, totally independent of whether or not one starts from initial conditions in the ground field.

Table 2 shows the first five exact iterates when fixing parameters at (a_α, b_α) and starting the iteration from the combination of zeros denoted by p in Fig. 4. Instead of small prime numbers raised to ever increasing powers, one observes the appear-

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Fig. 4. Basins of attraction for parameters corresponding to the multiple intersection of the three lowest possible periods. The basin of the attractor at infinity is indicated in black, yellow indicates the basin of the stable fix-points while the color purple (which appears to be embedded inside the yellow basin and to accumulate on the boundary of the black basin of infinity) indicates the basin for stable period-3 motion. u indicates the unstable fix-point while p and q are the two additional combinations of the zeros defining fix-points. The nine smaller dots indicate all possible combinations of the zeros obtained from the cubic (degenerate sextic), defined in Eqs. (23). The subset of three points indicated by the numbers $1 \to 2 \to 3 \to 1$ indicates the points belonging to the stable orbit of period-3. The "color" (periodicity) of the stable orbit living in the purple basin evolves continuously with changes in parameters, eventually "maturing" into black, i.e. with the region in purple turning into black and becoming part of the basin of $-\infty$.

ance of sequences of prime numbers of ever increasing sizes. We have also considered the sequence of exact iterates started from rationals relatively close to the point $p = (0.633975\ldots, -1.90192\ldots)$ like, for example, from the point $(x_0, y_0) = (63/100, -19/10)$. This case also generates large prime numbers. But the difference is that while in the first case one finds the denominator to be the number 2, the denominator for iterates started from rational approximations contain powers of 10 that increase rapidly. It is curious that apparently independently from the numbers used to start iterations, numerators seem to invariably involve huge prime numbers rather than combinations of relatively moderate primes raised to powers that would increase with the iteration.

To obtain a rough idea of the relative speed with which one approaches the fixed point when starting from p we performed iterations with quadruple precision, measuring the distance of the fixed point as iteration proceeds. After 10^5 iterates the distance is still 9.6×10^{-3}. Increasing the number of iterates to 10^6 reduced the distance to about 10^{-4}, meaning a rather low rate of convergence. From the specific law underlying the iteration it seems possible to show that in spite of starting from initial conditions given by zeros in the field, the infinite sequence of iterates is just asymptotic [not pre-periodic] to the periodic attractor although we have not been able to prove this.

We observe that our Fig. 4 has some similarity with Fig. 2 of Refs. [28,29] or with Fig. 1 of Ref. [30]. These references describe how a smooth basin boundary can become a fractal basin boundary when a system parameter is varied continuously. They describe a basin boundary "metamorphosis" in which the extent of a fractal basin can grow discontinuously by "suddenly sending a Cantor set of thin fingers into the territory of another basin" [29]. When (a) changing parameters smoothly enough and/or (b) changing more than one parameter simultaneously, we also observed the sudden appearance of additional basins in territory previously occupied by another basin. But the sudden appearance occurs apparently only for those specific parameters where cascades of some definite periodicity, say k, are born. For such parameters, fingers (more precisely, the whole additional basin that is then born) appear suddenly in place. Once this additional basin has appeared, further changes of parameters to the right of line II in the setup defined in Fig. 1 produce only a quite smooth evolution of the basin. As bifurcations proceed, the most abrupt phenomena that we observe are the familiar changes of the periodicity of the attractor living in the basin. The basin evolves rather continuously during the whole bifurcation process and also between changes among chaotic attractors. Only at the "end of the bifurcation cascade" [i.e. after the full alternation of all doubling cascades and all chaotic motions] is that the bounded attractor living inside the additional basin turns into an unbounded attractor. This means that the additional basin becomes now, in fact, part of the basin of the attractor at infinity. But rather than being an isolated and discontinuous phenomenon, the eventual "enlargement of the basin of infinity" seems to be the last step of a quite continuous evolution of an already pre-existing basin of an attractor living at finite distances.

For example, Fig. 4 shows precisely a situation where the sudden appearance of a new basin occurs. The basin in purple is the basin of a stable attractor of period three (indicated by the three green dots). The parameters $(a,b) = (a_\alpha, b_\alpha)$ corresponding to this figure lie by construction precisely on the line II discussed previously in Fig. 1. Any parameter variation to the left of line II will transform the stable orbit of period-3 existing over the reals into an orbit over the complex numbers. In other words, the basin in purple will suddenly change into yellow indicating that all initial conditions in the region that formerly constituted the period-3 purple basin must now converge asymptotically to the fixed point s. This because the period−3 orbit is no longer real. Moving smoothly to the right of line II one observes a continuous evolution of the purple basin. As parameters change, the period of the stable orbit that lives in the purple basin displays the usual doubling cascade $3 \to 6 \to 12 \to \cdots$ as indicated in Fig. 3. Eventually, beyond all familiar alternations of periodic and chaotic phenomena, one reaches parameters where the stable attractor corresponding to the purple basin is the fixed point at infinity. At this parameter, after displaying a full rainbow of colors, from the birth of the first to the death of the last attractor living in the (initially) purple basin, the basin changes from being a basin of an attractor located at finite distance into being the basin of the attractor located at infinity. From this parameter on the basin of infinity will indeed display a Cantor set of thin fingers. But they seem to originate from a rather continuous evolution of the purple basin, not from any sudden phenomenon.

The above observation is not necessarily in conflict with the sudden metamorphosis described in Refs. [28–30]: it might still be possible for fractal fingers to be sent abruptly into foreign territory, because in parameter space there is a profusion of parabolic arcs like the one corresponding to period-3 in Fig. 3, each of these new arcs having a period k and an associate cascade $k \times 2^n$. The accumulation points of these cascades need to be investigated with detail in order to understand what is really going on in these regions of parameter space which are characterized by the coexistence of several different stable motions that appear and disappear within small parameter intervals.

We have also observed that within specific parameter intervals it is rather easy to find additional basins of stable trajectories living *inside* the territory of the purple basin of period-3 (and inside the basins which evolve from this period-3 basin as the periodicity of the attractor in it evolves). As one comes closer and closer to accumulation points, one finds a sort of "basin splitting effect": every basin seems to contain embedded in it additional basins for intervals of parameters that are not easy to predict in advance but are certainly large enough to be numerically detected. A clear understanding of the simultaneous interplay of the many coexisting stable attractors certainly requires additional investigation.

7.3. *How close to attractors can we ever hope to get?*

It is clear that all possible physical attractors will be defined by numbers belonging to the ground field enforced by the parameters and by the irreducible equation characterizing the periodicity of the attractor. For polynomial equations of motion, all phase space dynamics corresponding to *periodic motions* will necessarily involve *algebraic dependence on parameters* and, therefore, "algebraic attractors", i.e. even if one chooses, say, a transcendental parameter, all attractors in phase space will be algebraic over the extended field. The reciprocities enforced by the dynamics will have the effect of making interesting points in parameter space to be necessarily defined by algebraic numbers very frequently. In practical applications we will always be forced to work with *approximations* to algebraic numbers and, therefore, an important question that needs to be considered in order to understand the physics is "how well can we approximate algebraic numbers?".

A quite elementary way of approximating algebraic numbers is by using rational numbers. As already remarked by Liouville in 1844, there is an obvious limit to the accuracy with which algebraic numbers can be approximated by rationals. This fact is a consequence of the definition of algebraic number [31]. If α is an algebraic number of degree $n \geq 2$ and p/q any rational approximation to it, then

$$|\alpha - p/q| > C/q^n, \tag{25}$$

where C is a positive constant depending only on α. By iterating further and further with absolute precision arithmetics it is possible to get arbitrarily close to algebraic numbers starting from rationals. But strictly, when starting from rationals and even from almost all numbers, algebraic or not, one will never be able to reach the precise ground field corresponding to an arbitrary periodic attractor with just a finite number of iterates, no mattering how large this number might be. *For almost all initial conditions it will be impossible to reach algebraic attractors precisely in a completely analogous way as it is totally impossible to actually reach attractors located at infinite distances with just a finite number of iterates.* What happens is that one remains "spiraling" indefinitely towards the attractor without ever reaching it, with the dynamics in phase space being ruled by ever increasing prime numbers. We recognize that already at this "classical level" of the dynamics there is a quite strong mechanism preventing one from reaching stable attractors from almost all initial conditions. These are simple consequences of the algebraic closure of the precise dynamics.

8. Conclusions

The basic new result reported in this paper is that points in parameter space defined by multiple intersections of boundary curves which delimit domains of macroscopically different physical behaviors are defined by specific mathematical entities: units. The familiar cascades of bifurcations imply the existence of an infinite quantity of points of accumulation of sequences of units in parameter space. Units corresponding to intersections of neighboring domains characterized *exclusively* by periodic behaviors need to be always defined by algebraic numbers. The existence of "mixed intersections", i.e. intersections which involve simultaneously domains of periodic and domains of chaotic behaviors seem to require the existence of a sort of "transcendental unit", i.e. a non-algebraic number that is a unit. In particular, *the accumulation points* (beyond which one finds chaos to be stable following every cascade) *mark precisely the transition* *from algebraic to non-algebraic units.* An interesting question now is whether or not it is possible to find analytical expressions allowing to generate all units belonging to a sequence and expressions allowing to move from one point of intersection to all subsequent ones. In particular, such formulas would be of interest to establish a sort of measure of the "proximity" of an accumulation point which after any *finite* number of steps will invariably look like a mirage, impossible to reach in practice.

We observed that by eliminating all variables but one, the dynamics of systems defined by polynomial equations of motion can be effectively reduced to the investigation of one-dimensional equivalent systems. These equivalent systems involve invariably certain prime factors which rule the dynamics by imposing specific ground fields for all attractors in phase space. To answer stability questions, rather than investigating all possible initial conditions, we observed that for each periodic motion it is enough to consider just the set of all possible combinations of the zeros of the prime factors defining the periodic orbit. The precise ground field cannot be reached from almost all initial conditions after just a finite number of iterates. Even when starting from initial conditions in the set of combinations of the zeros of the prime factor, it is necessary to use appropriate "in-phase" combinations in order to be able to reach attractors precisely.

In contrast with the familiar approach where one investigates stability questions by dealing with eigenvalue problems, we write the equations of motions in such a way as to incorporate a stability parameter m in them. In this way we could investigate more comfortably all solutions sharing common boundaries in parameter space by constructing and factoring the W_k surfaces as discussed in Section 5.

Units seem to pervade dynamical systems and certainly provide remarkably interesting parameter values where to probe with absolute numerical precision phase space dynamics of physical models and to address rather new questions, for example, the number-theoretic *origin* of periodicity, of aperiodicity and of the physical stability in general. So far, determinism has been almost invariably confused with effective computability. But it seems important to notice that from the mere existence of physical laws it does not follow automatically that the consequences of these laws can be always computed with sufficient accuracy, or even at all. It would be nice if the realization of the presence and important role in physics of relatively sophisticated mathematical entities and structures could prove beneficial for both fields: to understand the intricacies of the exact dynamics and to construct number-theoretic relations of interdependence among particular sets and lattices of numbers and to construct continuous functions with them and on them. The ability of predicting the future seems to depend critically on the precise number-theoretic knowledge of the numbers defining parameter values and initial conditions and on their relative commensurabilities and phase dependences.

It is certainly possible to use sequences of points (i.e. orbits generated by certain simple dynamical systems) to construct the basic functions of trigonometry, in essence the exponential function, similarly as done recently for the *arccos* function [6]. Exploiting certain peculiarly regular lattices of points produced by combining suitably the zeros of equations of motions as they appear generation after generation in certain dynamical systems one may also find, for specific choices of parameters, an alternative way of producing elliptic and continuous but non-differentiable functions as done by Weierstraß(1815–1897) and, according to Spalt [32], decades earlier by Bolzano (1781–1848). It would be nice to find a general framework based entirely on functions constructed directly from discrete cycles generated via dynamical systems and containing free "lattice-parameters" [defining the algebraic structure characterizing the dynamics] which could be "tuned" at will. Such functions would constitute rather privileged basis-functions to be used for expansions in physical theories.

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Appendix. Parameter symmetries defining ground fields

This appendix provides explicit expressions for the coefficients c_j and d_j appearing in Eqs. (11). These coefficients define the ground field for motions of period 3 and 4. From them one may recognize which symmetries are generated by the dynamics and enforce the appearance of units in parameter space.

All possible non-trivial period-3 orbits in phase space involve necessarily points which are suitable combinations of zeros of the following polynomial of degree 6:

$$p_3(a,b,x) = \sum_{j=0}^{6} c_j x^j, \tag{26}$$

where the coefficients c_j are given by

$$\begin{aligned}
c_0 &= -1 - b - b^2 - b^4 - b^5 - b^6 + a^3 - 2a^2(1+b^2) \\
&\quad + a(1 - 4b - 5b^2 - 4b^3 + b^4), \\
c_1 &= 1 + b + b^2 - b^3 - b^4 - b^5 + a^2(1-b) - 2a(1-b^3), \\
c_2 &= -1 + 2b + b^2 + 2b^3 - b^4 - 3a^2 + a(3 - 2b + 3b^2), \\
c_3 &= 1 - b + b^2 - b^3 - 2a(1-b), \\
c_4 &= -1 + 2b - b^2 + 3a, \\
c_5 &= 1 - b, \\
c_6 &= -1.
\end{aligned} \tag{27}$$

Similarly, all non-trivial period-4 orbits involve combinations of zeros of the following polynomial of degree 12:

$$p_4(a,b,x) = \sum_{j=0}^{12} d_j x^j, \tag{28}$$

where the coefficients d_j are given by

$$\begin{aligned}
d_0 &= 1 + 3b + 3b^2 + 4b^3 + 9b^4 + 9b^5 + 6b^6 + 9b^7 + 9b^8 \\
&\quad + 4b^9 + 3b^{10} + 3b^{11} + b^{12} + a^6 \\
&\quad + a^5(-3 - 2b - 3b^2) + a^4(3 + 4b + 2b^2 + 4b^3 + 3b^4) \\
&\quad + a^3(-3 - 4b + 7b^2 + 16b^3 + 7b^4 - 4b^5 - 3b^6) \\
&\quad + 2a^2(1 - b - 8b^2 - 7b^3 - 2b^4 - 7b^5 - 8b^6 - b^7 + b^8), \\
d_1 &= a^4(-1 - b + b^2 + b^3) + 2a^3(1 + 3b + 2b^2 - 2b^3 - 3b^4 - b^5) \\
&\quad + a^2(-1 - 5b - 9b^2 - 5b^3 + 5b^4 + 9b^5 + 5b^6 + b^7) \\
&\quad + 2a(1 + 4b + 5b^2 + b^3 - b^4 + b^5 - b^6 - 5b^7 - 4b^8 - b^9), \\
d_2 &= -6a^5 + 4a^4(3 + 2b + 3b^2) + 2a^3(-3 - 4b - 2b^2 - 4b^3 - 3b^4) \\
&\quad + a^2(5 + 4b - 13b^2 - 24b^3 - 13b^4 + 4b^5 + 5b^6) \\
&\quad + a(-1 + 2b + 8b^2 - 2b^3 - 14b^4 - 2b^5 + 8b^6 + 2b^7 - b^8), \\
d_3 &= -1 - 5b - 6b^2 + 2b^3 + 4b^4 - 4b^5 - 2b^6 + 6b^7 + 5b^8 + b^9 \\
&\quad + 4a^3(1 + b - b^2 - b^3) + 4a^2(-1 - 3b - 2b^2 + 2b^3 + 3b^4 + b^5), \\
d_4 &= 15a^4 - 6a^3(3 + 2b + 3b^2) + a^2(3 + 4b + 2b^2 + 4b^3 + 3b^4) \\
&\quad + a(-4 - 2b + 8b^2 + 12b^3 + 8b^4 - 2b^5 - 4b^6), \\
d_5 &= 6a^2(-1 - b + b^2 + b^3) + 2a(1 + 3b + 2b^2 - 2b^3 - 3b^4 - b^5), \\
d_6 &= 1 - b^2 - b^4 + b^6 - 20a^3 + 4a^2(3 + 2b + 3b^2), \\
d_7 &= 4a(1 + b - b^2 - b^3), \\
d_8 &= 15a^2 - a(3 + 2b + 3b^2), \\
d_9 &= -1 - b + b^2 + b^3, \\
d_{10} &= -6a, \\
d_{11} &= 0, \\
d_{12} &= 1.
\end{aligned} \tag{29}$$

As one recognizes without difficulty from the above expressions, all dependencies in one of the parameters, the parameter b here, appear invariably in the form of *reciprocal* polynomials. By introducing the simple change of variable $B \equiv b + 1/b$ one sees that whatever the numerical value of the zeros of the reciprocal polynomials, these zeros will necessarily be defined by units. A precious property resulting from the generation of reciprocal polynomials by the dynamics is that the mere *existence* of the transformation $B \equiv b + 1/b$ is enough to guarantee that even in those cases for which one could not possibly solve the polynomials in B analytically in terms of radicals, such solutions would still necessarily produce units.

By factoring b instead of a in the expressions for c_j and d_j one sees that although the parameters a are simply connected with units in the ground field, this fact is not so obvious to recognize from such alternative factorization. The most convenient way to proceed with the analytical work seems to be by seeking always to work with numbers and functions defined as linear combinations over the simplest possible field, $\mathbb{Z}$ in the present example (not $\mathbb{Q}$).

From the results presented in Section 5 we know that period-3 orbits are born along the line $W_3^+ = -4a + 7 + 10b + 7b^2 = 0$. Therefore, substituting the value of a defined by this equation into the sextic polynomial, Eq. (26) , one sees that along $W_3^+ = 0$ the sextic might be factored into the square of a cubic polynomial:

$$[1 + b - b^2 - b^3 - 18(1 + 2b + b^2)x - 4(1 - b)x^2 + 8x^3]^2 = 0. \tag{30}$$

The zeros of this equation are the points which define (degenerate) period-3 orbits living on $W_3^+ = 0$. In a perfect analogous way, there is a point along the variety $W_2^- \equiv -4a + 5 - 6b + 5b^2 = 0$ for which the polynomial of degree 12 factors into

$$p_4(a, b, x) = [1 - 2b + b^2 - 4(1 - b)x + 4x^2]^2 \sum_{j=0}^{8} e_j x^j \tag{31}$$

where

$$\begin{aligned}
e_0 &= 1021 + 2584b - 12852b^2 + 31784b^3 - 38674b^4 + 31784b^5 \\
&\quad -12852b^6 + 2584b^7 + 1021b^8, \\
e_1 &= 1672 + 4904b - 21944b^2 + 40360b^3 - 40360b^4 \\
&\quad +21944b^5 - 4904b^6 - 1672b^7, \\
e_2 &= 2336 - 5632b + 10720b^2 - 11776b^3 + 10720b^4 - 5632b^5 + 2336b^6, \\
e_3 &= 2912 - 13280b + 24768b^2 - 24768b^3 + 13280b^4 - 2912b^5, \\
e_4 &= -96 - 1152b + 1984b^2 - 1152b^3 - 96b^4, \\
e_5 &= -2688 + 5504b - 5504b^2 + 2688b^3, \\
e_6 &= 1024(-1 + b - b^2), \\
e_7 &= 512(1 - b), \\
e_8 &= 256.
\end{aligned} \tag{32}$$

Notice that in this case the quadratic factor in $p_4(a, b, x)$ coincides with the prime factor determining the ground field for orbits of period two.

A particularly interesting point in parameter space defining degenerate prime factors and orbits of period 4 is defined by the zero near $b_d \simeq -0.178101074\cdots$ of the equation

$$1 - 15b^2 + 64b^3 - 128b^4 + 164b^5 - 128b^6 + 64b^7 - 15b^8 + b^{10} = 0. \tag{33}$$

The corresponding value of a is $a_d = (5 - 6b_d + 5b_d^2)/4 \simeq 1.5568016033\cdots$.

References

[1] J.A.C. Gallas, A method for studying stability in physical models, Physica A 211 (1994) 57–83.

[2] E. Ott, Chaos in Dynamical Systems (Cambridge University Press, Cambridge, 1993).

[3] H.E. Nusse and J.A. Yorke, Dynamics: Numerical Experiments (Springer Verlag, New York, 1994).

[4] J.A.C. Gallas, C. Grebogi and J.A. Yorke, Vertices in parameter space: double crises which destroy chaotic attractors, Phys. Rev. Lett. 71 (1993) 1359–1362.

[5] I. Kan, H. Koçak and J.A. Yorke, Persistent homoclinic tangencies in the Hénon family, Physica D 83 (1995) 313–325.

[6] J.A.C. Gallas, Structure of the parameter space of a ring cavity, Applied Physics B 60 (1995) S203-S213, Festschrift H. Walther.

[7] D. Hilbert, Die Theorie der algebraischen Zahlkörper. Jahresb. der Deutschen Mathematiker-Vereinigung 4 (1897) 175–546 ≡ David Hilbert, Gesammelte Abhandlungen, Band I (Springer Verlag, Berlin, 1970).
[8] E. Hecke, Vorlesungen über die Theorie der algebraischen Zahlen (Akademische Verlagsgesellschaft, Leipzig, 1923) ≡ Lectures on the Theory of Algebraic Numbers (Springer Verlag, New York, 1981).
[9] M. Pohst and H. Zassenhaus, Algorithmic Algebraic Number Theory (Cambridge University Press, Cambridge, 1993).
[10] H. Cohn, Advanced Number Theory (Dover, NY, 1980) p. 112.
[11] K. Baclawski and M.D. Donsker, eds., Mark Kac: Probability, Number Theory, and Statistical Physics – Selected Papers (The MIT Press, Cambridge, 1979).
[12] M.R. Schroeder, Number Theory in Science and Communication, with applications in Cryptography, Physics, Biology, Digital Information, and Computing, Springer Series in Information Sciences (Springer Verlag, Berlin, 1984).
[13] J.-M. Luck, P. Moussa and M. Waldschmidt, eds., Number Theory and Physics, Springer Proceedings in Physics 47 (Springer Verlag, Berlin, 1990).
[14] M. Waldschmidt, P. Moussa, J.-M. Luck and C. Itzykson, eds., From Number Theory to Physics (Springer Verlag, Berlin, 1992).
[15] R.W.K. Odoni, The Galois theory of iterates and composites of polynomials, Proc. London Math. Soc. 51 (1985) 385–414.
[16] W. Narkiewicz, Polynomial cycles in algebraic number fields, Colloq. Math. 58 (1989) 151–155.
[17] M. Stoll, Galois groups over $\mathbb{Q}$ of some iterated polynomials, Arch. Math. 59 (1992) 239–244.
[18] F. Vivaldi, Dynamics over irreducible polynomials, Nonlinearity 5 (1992) 941–960.
[19] F. Vivaldi and S. Hatjispyros, Galois theory of periodic points of rational maps, Nonlinearity 5 (1992) 961–978.
[20] J.C. Lagarias, Number theory and dynamical systems, Proceedings of Symp. Appl. Math. 46 (1992) 35–72.
[21] P. Morton, Arithmetic properties of periodic points of quadratic maps, Acta Arithmetica 62 (1992) 343–372.
[22] A. Batra and P. Morton, Algebraic dynamics of polynomial maps on the algebraic closure of a finite field, I, Rocky Mountain J. Math. 24 (1994) 453–481.
[23] A. Batra and P. Morton, Algebraic dynamics of polynomial maps on the algebraic closure of a finite field, II, Rocky Mountain J. Math. 24 (1994) 905–932.
[24] P. Morton and P. Patel, The Galois theory of periodic points of polynomial maps, Proc. London Math. Soc. 68 (1994) 225–263.
[25] J.A.C. Gallas, Structure of the parameter space of the Hénon map, Phys. Rev. Lett. 70 (1993) 2714–2717.
[26] J.A.C. Gallas, Dissecting shrimps: Results for some one-dimensional physical models, Physica A 202 (1994) 196–223.
[27] B.L. van der Waerden, Algebra, Bd. 1, 9. Auflage, S. 103 (Springer Verlag, Berlin, 1993).
[28] C. Grebogi, E. Ott and J.A. Yorke, Metamorphoses of basin boundaries in nonlinear dynamical systems, Phys. Rev. Lett. 56 (1986) 1011–1014.
[29] C. Grebogi, E. Ott and J.A. Yorke, Basin boundary metamorphoses: changes in accessible boundary orbits, Physica D 24 (1987) 243–262.
[30] K.T. Alligood, L. Todeschini-Lalli and J.A. Yorke, Metamorphoses: sudden jumps in basin boundaries, Commun. Math. Phys. 141 (1991) 1–8.
[31] A. Baker, Transcendental Number Theory, (Cambridge University Press, Cambridge, 1990).
[32] D.D. Spalt, Buchbesprechung über die Bernhard Bolzano Gesamtausgabe, Jahresb. der Deutschen Mathematiker-Vereinigung 96 (1995) 88–91.