Physica Scripta. Vol. T118, 143–147, 2005

Inducing Periodicity in Lattices of Chaotic Maps with Advection

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Received October 18, 2003; revised version received July 26, 2004; accepted August 13, 2004

PACS numbers: 89.75.Kd, 05.45.Xt, 05.45.Ra

Abstract

We investigate a lattice of coupled logistic maps where, in addition to the usual diffusive coupling, an advection term parameterized by an asymmetry in the coupling is introduced. The advection term induces periodic behavior on a significant number of non-periodic solutions of the purely diffusive case. Our results are based on the characteristic exponents for such systems, namely the mean Lyapunov exponent and the co-moving Lyapunov exponent. In addition, we study how to deal with more complex phenomena in which the advective velocity may vary from site to site. In particular, we observe wave-like pulses to appear and disappear intermittently whenever the advection is spatially inhomogeneous.

1. Introduction

Recent studies of pattern formation and pattern dynamics in spatially extended systems have provided important clues to understand nonlinear mechanisms of nonequilibrium conditions in many physical phenomena such as, e.g., laser dynamics [1], electroconvection [2], Rayleigh-Bénard convection [3], field-induced phenomena in magnetic fluids [4], and many others [5]. A very popular way to study pattern formation is by using networks of coupled oscillators ruled locally by time-discrete mappings, the so-called coupled map lattices [6]. Such networks of maps were applied e.g. to model information coding [7], to study nonlinear wave-like patterns [8], ocean convection parameterization [9] and synchronization processes [10, 11].

Usually, coupled map lattices are regarded as a discretize version of reaction-diffusion systems [6], because they involve a set of discrete-time nonlinear oscillators coupled through diffusion alone. Pattern formation in networks of maps was already studied for this purely diffusive regime [6, 12]. However, as is well-known [13], spatially extended phenomena are quite frequently subject not only to diffusion but also to advection. Denoting by γ_i the advection strength at site i, a simple way of incorporating advection in networks of maps was proposed recently [14] being embodied in the following equation of motion

$$x_{t+1}(i) = f(x_t(i)) + \varepsilon\mathcal{D}_{i,t} - \gamma_i\mathcal{A}_{i,t}, \tag{1}$$

where $i = 1, \ldots, L$, with L being the lattice size and ε represents the diffusion. Following common practice, we also use the logistic map, $f(x) = 1 - ax^2$, to drive the local dynamics. Here $\mathcal{D}_{i,t}$ and $\mathcal{A}_{i,t}$ are discretized forms of the diffusion and advection operators, namely

$$\begin{aligned}\mathcal{D}_{i,t} &= \frac{f(x_t(i+1)) + f(x_t(i-1))}{2} - f(x_t(i)),\\ \mathcal{A}_{i,t} &= \frac{f(x_t(i+1)) - f(x_t(i-1))}{2}.\end{aligned} \tag{2}$$

As shown in Ref. [8], γ_i must be in the interval $-\varepsilon \le \gamma_i \le \varepsilon$. We impose periodic boundary condition: $x_t(L+1) \equiv x_t(1)$.

For $\gamma = 0$ Eq. (1) reduces to the well-known purely diffusive model, while for $\gamma = \pm\varepsilon$ one obtains the two possible one-way coupling regimes [6]. In the presence of advection one may distinguish two different situations: *homogeneous advection*, when $\gamma_i \equiv \gamma$ for all sites i, and *inhomogeneous advection*, when the γ_i are free to vary along the lattice. In an earlier investigation [14] we described how the spatial periodicity (wavelength) of wave-like patterns evolve with the advection strength. The purpose of this paper is to show how advection affects the *temporal* stability of patterns. In particular, we address the question of how and under which conditions advection changes pattern evolutions from chaotic to periodic evolutions.

The stability of solutions in dynamical systems is established by the well-known Lyapunov analysis, considering the so-called local Lyapunov exponents, defined from the logarithm of the eigenvalues of the system. If the evolution is periodic one finds a negative exponent while for chaotic evolutions the exponent is positive.

Local Lyapunov exponents are computed in a static frame. However, recently [6, 15] it was shown that, for particular situations such as one-way coupling regimes, unstable perturbations may travel along the lattice, with a corresponding *negative* local Lyapunov exponents. Therefore, local Lyapunov exponents may not be always suitable to characterize the stability of such solutions. To cure this shortcoming one uses the so-called co-moving Lyapunov exponents [6, 15].

Co-moving Lyapunov exponents are defined from the eigenvalues of a Jacobian computed in a moving frame having some 'velocity'. The most important feature of co-moving exponents lies in the fact that they allow to discriminate between (*i*) absolute stability, (*ii*) absolute instability and (*iii*) convective instability, which is a possible feature in spatially extended systems. Furthermore, with co-moving Lyapunov exponents one may study, e.g., transitions from regular patterns to spatiotemporal intermittency [15], propagation of correlations [16], and predictability [17].

In this paper we study the stability of pattern evolutions using both local and co-moving Lyapunov exponents. In Section 2 we start with a situation where advection is homogeneously distributed. In Section 3 we consider nontrivial behaviors of pattern evolutions when advection varies in space. Final conclusions are given in Section 4.

2. Switching non-periodic into periodic evolutions

The purpose of this Section is to study the influence of diffusion and homogeneous advection in pattern evolutions when ruled

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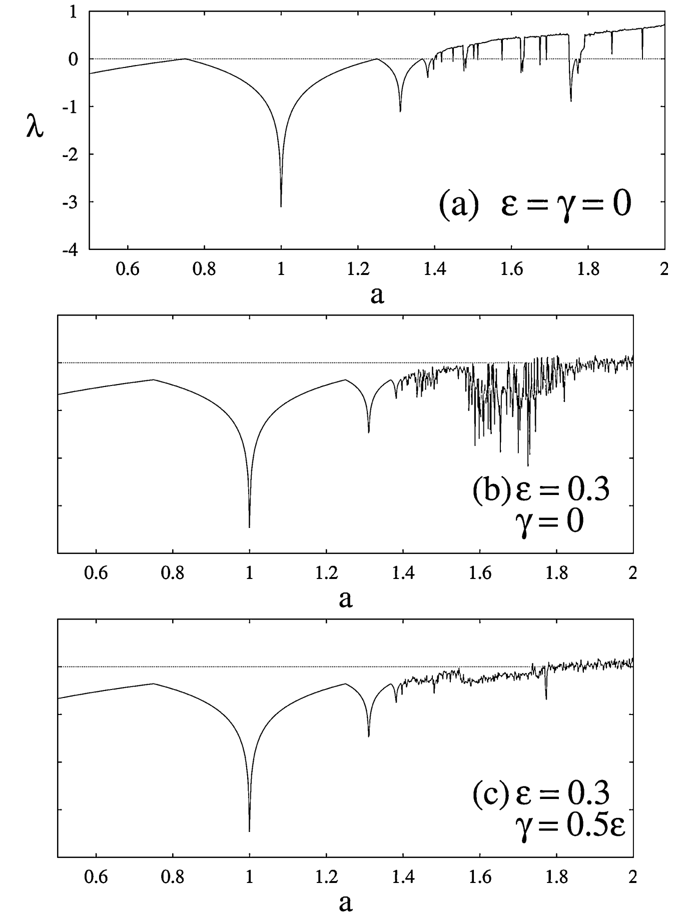

Fig. 1. Local Lyapunov exponent λ as a function of local nonlinearity a, (a) for the uncoupled regime ($\varepsilon = 0$ and $\gamma = 0$), (b) for the purely diffusive regime ($\gamma = 0$) with $\varepsilon = 0.3$, and (c) for $\varepsilon = 0.3$ and $\gamma = 0.5\varepsilon$. Here $L = 64$.

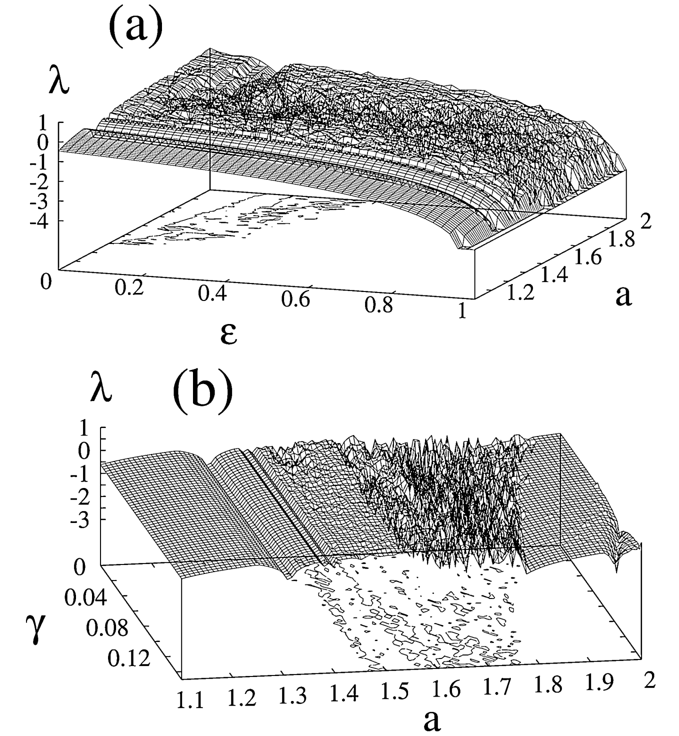

Fig. 2. Variation of the Local Lyapunov spectrum as a function of the local nonlinearity a with (a) the diffusion ε for $\gamma = 0$, and (b) the advection γ for $\varepsilon = 0.15$. Here $L = 64$ and contours indicate the regions where Lyapunov exponents are $\lambda = 0$.

by Eq. (1) with $\gamma_i \equiv \gamma$. We start computing the local Lyapunov exponents, given by the logarithm of the maximum absolute value of all eigenvalues of the Jacobian of Eq. (1).

For the uncoupled regime ($\varepsilon = 0$ and $\gamma = 0$), local Lyapunov exponents reduce to the Lyapunov exponents of the local map, as illustrated in Fig. 1a. We use a sample of 100 sets of initial conditions of the form $x_0(i) = x^* + \delta\theta_r(i)$ where $x^* = -(1 + \sqrt{1 + 4a})/2$ is the unstable fixed point of the local logistic map, $\theta_r(i)$ is a homogeneously distributed random value in the range $[0, 1]$ and $\delta = 0.001$. The Jacobian of Eq. (1) is computed over 100 time-steps after discarding a transient of 100.000 time-steps. For purely diffusive coupling ($\varepsilon \neq 0$) one knows [6] that local Lyapunov exponents decrease. This fact is illustrated in Fig. 1b, where $\varepsilon = 0.3$, showing a Lyapunov spectrum which, before the accumulation point $a = 1.401155\ldots$ [12], has a similar shape as that of the uncoupled regime. Above the accumulation point one observes a much larger exponent fluctuation then that of the uncoupled regime.

Figure 1c shows, for the same conditions of Fig. 1a and 1b, the local Lyapunov exponent as a function of the nonlinearity when diffusion and advection are simultaneously present, namely $\varepsilon = 0.3$ and $\gamma = 0.5\varepsilon$. Comparing Figs. 1b and 1c one concludes that, apparently, advection does not change significantly the local Lyapunov spectrum. The main difference between the presence and absence of advection is observed above the accumulation point, where the large fluctuations observed in the purely diffusive regime are then shortened.

In general, these observations remain valid for any diffusion and advection strengths. For either periodic or chaotic local dynamics [6], our simulations have shown that diffusion promotes the stabilization of local amplitudes. Numerical fitting of the Lyapunov exponents show the ε-dependence to be quadratic, i.e. $\lambda \propto \sqrt{1 - \varepsilon}$.

Figure 2a shows the local Lyapunov exponent as a function of both local nonlinearity and diffusion strength. Here we use the same conditions of Fig. 1 and plot 100 values of a in the range $1 \leq a \leq 2$ for 50 uniformly distributed values of diffusion strength in the range $0 \leq \varepsilon \leq 1$. The contour lines under the figure denote loci where local Lyapunov exponents are zero. From this figure one clearly sees that diffusion promotes the stability of local oscillators. On the other hand, Fig. 2b shows the dependence on both the local nonlinearity and advection strength, illustrating that the local Lyapunov spectrum does not change significantly when advection is varied maintaining ε fixed. This behavior is typical for other values of ε.

As mentioned in the previous section, from local Lyapunov exponents one cannot distinguish between absolute and convective unstable solutions. For negative local Lyapunov exponents there is the possibility of having a convectively unstable pattern where propagation of perturbations are observed [6].

Next, we study the co-moving Lyapunov spectrum of pattern evolutions, determining the co-moving Lyapunov exponents from the definition of Deissler and Kaneko [18] (see Ref. [6] for details): the logarithm of the maximum eigenvalue of the matrix $\mathbf{J}(V)$, defined by

$$\mathbf{J}(V) = \prod_{t=1}^{T} \begin{bmatrix} \dfrac{\partial x_{t+1}^{1+[V(t+1)]}}{\partial x_t^{1+[Vt]}} & \cdots & \dfrac{\partial x_{t+1}^{1+[V(t+1)]}}{\partial x_t^{N+[Vt]}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_{t+1}^{N+[V(t+1)]}}{\partial x_t^{1+[Vt]}} & \cdots & \dfrac{\partial x_{t+1}^{N+[V(t+1)]}}{\partial x_t^{N+[Vt]}} \end{bmatrix}, \tag{3}$$

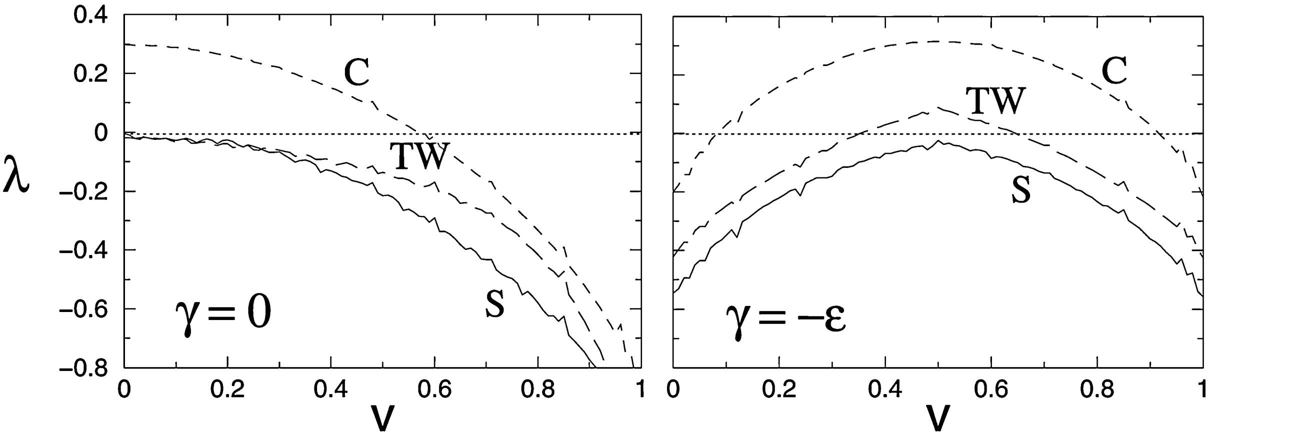

Fig. 3. Co-moving Lyapunov exponents, as a function of the velocity of the moving frame, for two extreme regimes: the purely diffusive regime ($\gamma = 0$) and the one way coupling regime ($\gamma = -\varepsilon$). Illustrative examples are shown, namely a chaotic pattern evolution (*C*) with $a = 1.9$, a static pattern evolution (*S*) for $a = 1.7$, and a traveling wave (*TW*) for $a = 1.73$. Here $\varepsilon = 0.5$ and $L = 100$.

where V is the velocity of the frame in which the Lyapunov exponent is determined, $T \gg 1$ is the number of time-steps used for the computation. In Eq. (3) we used the notation x_p^q, where p represents time and q represents *space* and, in both indices $[z]$ represents the integer part of z, and $N \le L$ represents the number of consecutive sites on the lattice followed by the frame. Notice that at each time-step the matrix $\mathbf{J}(V)$ operates between sites $\{1 + [Vt], \ldots, N + [Vt]\}$ and sites $\{1 + [V(t+1)], \ldots, N + [V(t+1)]\}$. For $10 \le N \le 20$, we computed the matrix $\mathbf{J}(V)$ during $T = 15000$ time-steps and determined the mean value of of the maximum eigenvalue.

Figure 3 shows the co-moving Lyapunov exponent for typical examples of chaotic pattern evolutions (C) and for periodic pattern evolutions, namely static patterns (S) and traveling waves (TW). Two extreme regimes are illustrated: the purely diffusive regime ($\gamma = 0$) and the one-way coupling regime $\gamma = -\varepsilon$.

In the purely diffusive regime ($\gamma = 0$), one clearly sees that chaotic pattern evolutions are absolutely unstable, i.e. the local Lyapunov exponent ($V = 0$) is positive, while both periodic pattern evolutions are absolutely stable, i.e. co-moving Lyapunov exponents are negative for all values of the frame velocity V. For one-way coupling the static pattern evolution remains absolutely stable while both the traveling wave and the chaotic pattern evolution become conditionally unstable, since there is a range of frame velocities for which one finds positive Lyapunov exponents.

To study the transition between these two extreme regimes one needs to compute co-moving Lyapunov exponents also as a function of the advection γ. Figure 4 shows the co-moving Lyapunov exponent for traveling waves solutions (Fig. 4a) and for chaotic pattern evolutions (Fig. 4b) when advection is increased from the purely diffusive regime to the one-way coupling regime. The contour lines under these figures indicate the boundaries where the co-moving Lyapunov exponent is zero. Static evolutions are always absolutely stable when γ varies from 0 to $-\varepsilon$, and therefore are not shown.

For traveling waves one observes a gradual increase of the co-moving Lyapunov exponents reaching a convectively unstable state for very strong advection ($\gamma \sim |\varepsilon|$), as shown in Fig. 4a.

For chaotic pattern evolutions a surprising effect is observed: by tuning properly the advection parameter it is possible to stabilize unstable chaotic pattern evolutions. In fact, from Fig. 4b one clearly observes two narrow ranges of advection strengths for which *absolute stability* is observed, i.e. all co-moving Lyapunov exponents become negative. In other words, the chaotic

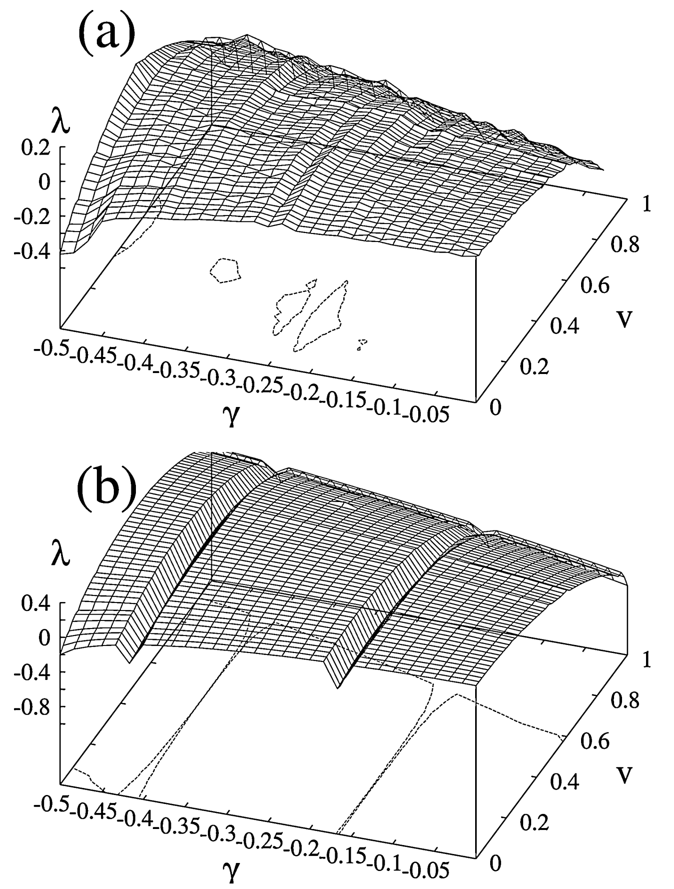

Fig. 4. (a) For traveling waves, co-moving Lyapunov exponents increase with the advection strength, and absolutely stable states evolve toward convectively unstable states. (b) For chaotic pattern evolutions, one observes an abrupt decrease of all co-moving Lyapunov exponents, when advection strength is increased. This decrease indicates that, for certain ranges of the advection strength, it is possible to change chaotic into periodic evolutions (see text). Here the same conditions as in Fig. 3 are used.

evolution abruptly starts to evolve periodically for these ranges of advection. This observation strongly suggest that, in fact, *advection may induce a sort of synchronization in a chaotic ensemble of nonlinear oscillators*. Notice that the nonlinearity of the local map is strongly chaotic, namely one has $a = 1.9$.

3. Effects of an inhomogeneous advection field

In this Section we compare the results above, obtained for uniform advection, with those for a more realistic and complicated

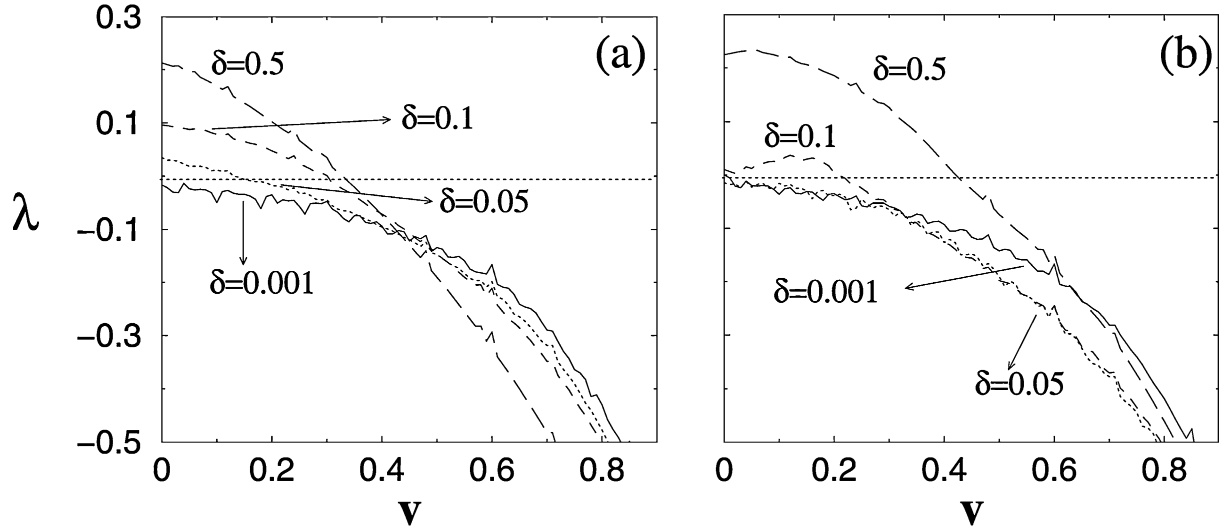

Fig. 5. When advection varies in space the co-moving Lyapunov spectrum as a function of the frame velocity increases with the width δ where advection γ is randomly distributed (see text). This occurs either (a) when advection direction remains unchanged, and only its strength is varied, and (b) when both strength and direction vary randomly. Here we choose a traveling wave solution, for $\gamma = 0$.

situation in which the strength and direction of advection vary from site to site. To this end, for each site i we choose a value $\gamma_i = r_i\delta\varepsilon$, where r_i are random numbers homogeneously distributed between -1 and 1 and δ measures the range where advection strength may fluctuate, i.e. measures the 'inhomogeneous fluctuation', yielding $-\delta\varepsilon \le \gamma \le \delta\varepsilon$, with $0 \le \delta \le 1$. For $\delta = 0$ we are back to the purely diffusive regime, while for $\delta = 1$ one has maximum range of variation.

Figure 5 shows the co-moving Lyapunov spectrum of a traveling wave solution, for an inhomogeneous advection with four different values of δ, namely for $\delta = 0.001, 0.05, 0.1, 0.5$. In Fig. 5a the advection strength varies in space, but its direction (sign) remains unchanged, namely values of γ are always positive, while in Fig. 5b both the strength and the direction of advection vary randomly in space.

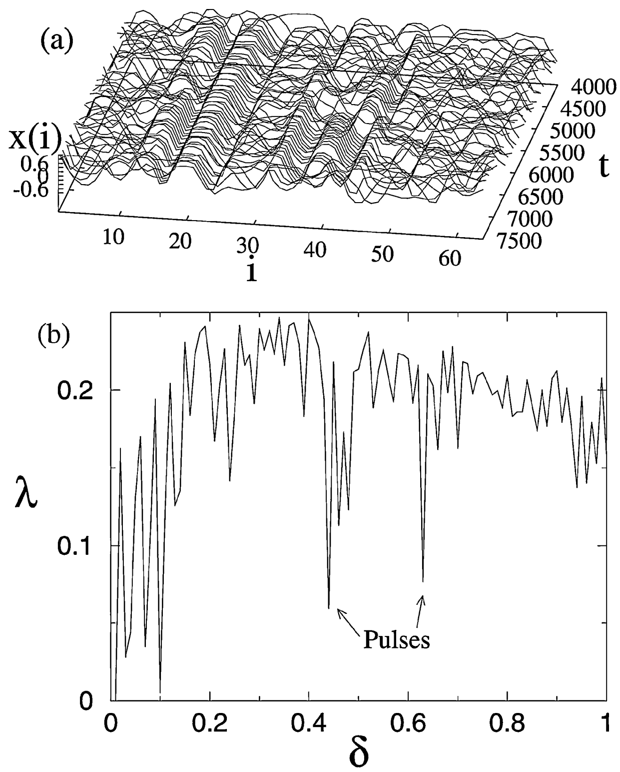

Fig. 6. Characterization of pulses in inhomogeneous advective lattices, through co-moving Lyapunov exponents. (a) Space-time diagram illustrating a typical pattern evolution observed in inhomogeneous advective lattices showing the appearance and disappearance of pulses (see text). (b) The maximum Lyapunov exponent as a function of the width δ of the interval $[-\delta\varepsilon, \delta\varepsilon]$ where γ_i is randomly distributed.

Comparing Fig. 5a and Fig. 5b with the Lyapunov spectrum of the traveling wave solution in Fig. 3, one clearly sees that the maximum co-moving Lyapunov exponents increases with δ, either when the direction of advection changes or not. As illustrated in Fig. 6a, this increase of the Lyapunov exponents is related to an increase of spatial disorder in the spatiotemporal diagram of the pattern evolution. In fact, when inhomogeneous advection is introduced, the traveling wave cannot maintain its wave-like shape through the entire lattice. Instead, only certain small domains maintain their wave-like shape. These domains may either remain static in an 'environment' which evolves chaotically, or appear and disappear intermittently. These latter domains which appear and disappear, we call pulses.

The occurrence of such pulses depends on the amplitude δ of the interval where advection is varied. Figure 6b shows the maximum co-moving Lyapunov exponent as a function of δ, indicating two particular values for which pulses are observed. As one clearly sees, pulses are characterized by an abrupt decrease of the maximum co-moving Lyapunov exponent, and this feature seems to prevail when varying the set of initial conditions and the time interval during which Lyapunov exponents are computed.

4. Conclusions

The purpose of this manuscript is to use local and co-moving Lyapunov exponents to characterize both periodic and aperiodic pattern evolutions in diffusive-advective coupled map lattices as defined in Ref. [14]. The effects of advection in the evolution of patterns were studied for both homogeneous and inhomogeneous advection fields.

The main result is that there are specific ranges of advection strengths which induce chaotic pattern evolutions to evolve periodically. This is observed from the fact that for those particular advection ranges all co-moving Lyapunov exponents turn out to be negative, i.e. the pattern evolution changes from an absolutely unstable state to an absolutely stable state.

For the particular case of one-way coupling regime, while the static pattern evolution remains absolutely stable, both traveling waves and chaotic pattern evolutions become conditionally unstable. Moreover, we showed that diffusion promotes the stability of local oscillators, corroborating previous studies [6]. For homogeneous advection, our simulations have shown that variation of N in Eq. (3) does not change significantly the final value of the co-moving Lyapunov exponent.

For an inhomogeneous advection field, traveling wave solutions are 'destroyed', and only small domains in the lattice keep their wave-like shape. These domains may appear and disappear intermittently corresponding to an abrupt decrease of the maximum co-moving Lyapunov exponent.

For inhomogeneous advection, the co-moving Lyapunov exponent varies with the size N of the matrix in Eq. (3). In fact, preliminary results have shown that there is a pair of values of the matrix size N and frame velocity V for which the co-moving Lyapunov exponent reaches a minimum. Therefore, we believe that by varying the value of N one should be able to ascertain when and where the pulses emerge in the lattice. This interesting characterization together with a few additional issues will be presented elsewhere.

Acknowledgments

PGL thanks *Fundação para a Ciência e a Tecnologia*, Portugal, for a doctoral fellowship. JACG is a CNPq Research Fellow, Brazil.

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