Delayed Choice Between Purely Classical States

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It is argued that Wheeler's insightful idea of delayed choice experiments may be explored at a classical level, arising naturally from number-theoretical conjugacies always necessarily present in the equations of motion. For simple and representative systems, we illustrate how to cast the equations of motion in a form encoding all classical states simultaneously through a "state parameter". By suitably selecting the parameter one may project the system into any desired classical state.

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The investigation of the relationship between quantum and classical states is experiencing nowadays a renewed upsurge of interest driven by new pressing challenges related to technological needs arising from quantum engineering [1, 2, 3, 4, 5, 6, 7] and from quantum information processors which promise efficient solution to problems that seem intractable using classical devices [8, 9]. A recent issue of Nature is dedicated to these novelties where it is explained why, "despite some remaining hurdles, the mind-bending and frankly weird world of quantum computers is surprisingly close" [10].

A particularly fruitful bridge facilitating the understanding of correlations between quantum and classical phenomena has been Bohr's principle of complementarity [11] stating that quantum systems, or "quantons" [12, 13] posses properties that are equally real but mutually exclusive. With this concept, the familiar wave-particle duality may be phrased more objectively as follows: Depending on the experimental circumstance a quanton behaves approximately either as a classical particle or as a classical wave. The standard way of exploiting quantons is by using interferometers such as the traditional double-slit experiment of Young [14], specially as manifest in the novel and very ingenious recent implementations [15, 16, 17], or the Mach-Zehnder setup. In these frameworks, the signature of wavelike behavior is the familiar interference pattern, whereas the signature of particlelike behavior emerges whenever one can discriminate along which way the interferometer has been traversed. Many interesting "classical effects" of interference have been discussed in the literature [18, 19, 20]. Inspired by a series of experiments proposed by Wheeler some years ago, we wish to consider here another type of classical analogon of notions familiar from investigations of quantons.

In an insightful contribution Wheeler named, spelled out and elucidated by seven examples the so-called "delayed choice experiments", instigated by the following question concerning the screen with two slits [21, 22]: Can one choose whether the photon (or electron) shall have come through both of the slits, or only one of them, after it has *already* transversed this screen? His motivation to ask this question was what he refers to as a *pregnant sentence* of Bohr: "...it... can make no difference, as regards observable effects obtainable by a definite experimental arrangement, whether our plans for constructing or handling the instruments are fixed beforehand or whether we prefer to postpone the completion of our planning until a later moment when the particle is already on its way from one instrument to another" [23]. Experimental confirmation followed soon [24, 25].

Here we argue that the fruitful concept of delayed choice is no alien to classical physics and may be directly recognized and retrieved from the multivalued nature of the equations of motion of all discrete-time dynamical systems of algebraic origin [26]. In classical dynamical systems, delayed choice is tantamount to orbital parameterization and arises thanks to number-theoretical properties shared by the equations defining the infinite set of periodic orbits building the scaffolding of orbits known to underly both classical and quantum dynamics[27, 28, 29, 30].

To illustrate how delayed choice works in classical systems we first consider the paradigmatic logistic map [31] $x \mapsto f(x) = a - x^2$. For $a = 4$ this system serves adequately for dynamics based computation, in particular to emulate logic gates, to encode numbers, and to perform specific arithmetic operations such as addition and multiplication on these numbers [32]. We consider a fully general situation however, letting a to be a free parameter. In discrete-time dynamics, the period is the foremost discrete ("quantized") quantity [33]. To illustrate concepts and methodology, we first derive a pair of polynomials, $P_3(x)$ and $S_3(\sigma)$, encoding simultaneously all possible orbits of period-3. Period 3 is the quintessence period in dynamical systems. For, existence of period 3 implies existence of all other periods because of the powerful theorem "period three implies chaos" [34].

Period-3 orbits are obtained by composing the equation of motion $f(x)$ three times consecutively, $f^{(3)}(x) \equiv f(f(f(x)))$, and isolating lower-lying periods by division as usual [33]. This yields the basic polynomial $H_3(x)$ with roots x_i ruling all possible period-3 motions

$$\begin{aligned} H_3(x) &= \big(x - f^{(3)}(x)\big)/\big(x - f(x)\big) \\ &= x^6 - x^5 - (3a-1)x^4 + (2a-1)x^3 \end{aligned}$$

$$+(3a^2-3a+1)x^2-(a-1)^2x$$
$$-a^3+2a^2-a+1. \qquad (1)$$

The degree of $H_3(x)$ reveals that we have to deal with two period-3 orbits, which for arbitrary a are distinct. So, generically we deal with two independent triplets of numbers which are entangled among the roots of $H_3(x)$. Although the coefficients of the sextic are relatively tame functions of a, its roots are not. From the infinity of possible relative-sextic quantities, phase-space dynamics knows precisely which adequate combination of algebraically *conjugate* numbers to select in order to produce tame $H_3(x)$ polynomials. The task now is to to find a means of efficiently disentangling the pair of triples composing individual period-3 orbits, i.e. of decomposing the sextic into a pair of cubics. This will be done now in two different ways. The first is a procedure that is sometimes practical. The other one works systematically.

Symmetric representation: The standard mathematical way of finding roots of polynomials relies on computing *discriminants* [35], generalizing the well-known procedure to solve quadratic equations. So, the discriminant of $H_3(x)$ is $\Delta_3 \equiv (4a-7)^3\left(16a^2-4a+7\right)^2$. It contains a factor which is not a perfect-square, thus a natural candidate for an extension field over which to attempt to factor the equation of motion. Indeed, introducing the radical $r \equiv (4a-7)^{1/2}$, over the relative-quadratic extension $\mathbb{Q}(r)$ of the rationals one finds a symmetric decomposition $H_3(x) = \psi_1(x)\psi_2(x)$, valid for arbitrary values of a, where

$$\psi_1(x) = x^3-ux^2-(a+v)x+1-av,$$
$$\psi_2(x) = x^3-vx^2-(a+u)x+1-au,$$

and where $u \equiv (1+r)/2$, $v \equiv (1-r)/2$. This symmetric pair of cubics expresses the orbits as conjugate factors of the radical r. Interchanging the branches of r simply converts one orbit into the other. These cubics already show the interesting property that we wish to explore: the *individual formal representation* of either $\psi_1(x)$ or $\psi_2(x)$ already contains in its structure information concerning all possible physical solutions. Particular solutions emerge only when we fix the branches of the radical r. The above derivation is not helpful in general because the presence of non-quadratic factors in the discriminant does not automatically imply factorizability. A method that works in general, for arbitrary periods, is the following.

Asymmetric representation: Independently of discriminants, the individual orbits entangled in $H_3(x)$ may be sorted out systematically as follows. Denote by ξ any arbitrary root of $H_3(x)$. To form a period-3 orbit, such root must be obviously connected to two companion roots: ξ, $f(\xi)$, $f^{(2)}(\xi)=f(f(\xi))$. These orbital points split $H_3(x)$ into cubics. They may be used to construct the familiar trio of elementary symmetric functions

$$\theta_1(\xi) = \xi+f(\xi)+f^{(2)}(\xi) \qquad (2a)$$
$$\theta_2(\xi) = \xi f(\xi)+\xi f^{(2)}(\xi)+f(\xi)f^{(2)}(\xi) \qquad (2b)$$
$$\theta_3(\xi) = \xi f(\xi)f^{(2)}(\xi), \qquad (2c)$$

which remain invariant under permutations of the orbital points. The fact that $f(\xi)=a-\xi^2$, $f^{(2)}(\xi)=a-(a-\xi^2)^2$ and that ξ is a root of $H_3(x)$ allows us to express any pair $\theta_m(\xi)$, $\theta_n(\xi)$ in terms of the remaining member of the trio. A fruitful choice is to express $\theta_3(\xi)$ and $\theta_2(\xi)$ in terms of the sum $\theta_1(\xi)$ of orbital points:

$$\theta_1(\xi) = -\xi^4+(2a-1)\xi^2+\xi+2a-a^2$$
$$\theta_2(\xi) = H_3(\xi)+\theta_1(\xi)+a+1 = \theta_1(\xi)+a+1$$
$$\theta_3(\xi) = (\xi+1)H_3(\xi)-a\,\theta_1(\xi)+a-1$$
$$= -a\,\theta_1(\xi)+a-1.$$

These three symmetric functions define the key cubic

$$P_3(x) = x^3-\theta_1(\xi)\,x^2+\big(\theta_1(\xi)+a+1\big)\,x$$
$$+a\,\theta_1(\xi)-a+1, \qquad (3)$$

the equation of motion for the disentangled orbit.

Similarly, calling η, $f(\eta)$ and $f^{(2)}(\eta)=f(f(\eta))$ the remaining triplet of roots of $H_3(x)$, one sees that they obey the same functional relations above, namely $\theta_1(\eta)$, $\theta_2(\eta)$ and $\theta_3(\eta)$. As already mentioned, this triplet is in general distinct from $\theta_1(\xi)$, $\theta_2(\xi)$ and $\theta_3(\xi)$, since they define different orbits.

Denoting indistinctly by $x_1, x_2, \ldots, x_6$ the roots of $H_3(x)$, the sum and product of $\theta_1(\xi)$ and $\theta_1(\eta)$ are then

$$\theta_1(\xi)+\theta_1(\eta) = \sum x_j = 1,$$
$$\theta_1(\xi)\,\theta_1(\eta) = \sum_{j<k} x_jx_k+\theta_2(\xi)+\theta_2(\eta)$$
$$= -(3a+1)+\theta_1(\xi)+\theta_1(\eta)+2(a+1)$$
$$= 2-a.$$

These two quantities define a quadratic, say $w^2-w+2-a=0$, with roots $w=(1\pm\sqrt{4a-7})/2$. They are the numbers $\theta_1(\xi)$ and $\theta_1(\eta)$ needed in Eq. (3) to obtain the pair of period-3 orbits. Instead of w we introduce a more convenient parameter σ through the transformation $4a-7=(2\sigma-1)^2$ or, equivalently, $S_3(\sigma)=0$ where

$$S_3(\sigma) = \sigma^2-\sigma+2-a, \qquad (4)$$

a polynomial that coincides with the above polynomial in w. The solutions of $w^2-w-a+2=0$ may be also written as $w=(1\pm 2\sigma\mp 1)/2$, yielding the final answers in a very convenient form:

$$\theta_1(\xi)=\sigma \qquad \text{and} \qquad \theta_1(\eta)=1-\sigma, \qquad (5)$$

or $\theta_1(\xi)=1-\sigma$ and $\theta_1(\eta)=\sigma$. Recalling Eq. (2a) one sees that, for each periodic orbit, the convenient parameter σ is simply the *sum* of its orbital points. Using the constraint $S_3(\sigma)=0$ we may eliminate a from Eq. (3) and obtain an equation whose unknown coefficient is either $\theta_1(\xi)$ or $\theta_1(\eta)$, depending which orbit we want to consider. For the choice in Eq. (5) we get

$$\varphi_1(x) \equiv \varphi_1(x;\sigma) = x^3-\sigma\,x^2-(\sigma^2-2\sigma+3)\,x$$

$$
\begin{aligned}
&\quad +\sigma^3 - 2\sigma^2 + 3\sigma - 1,\\
\varphi_2(x) \equiv \varphi_2(x;\sigma) &= x^3 - (1-\sigma)\,x^2 - (\sigma^2+2)\,x\\
&\quad -\sigma^3 + \sigma^2 - 2\sigma + 1,
\end{aligned}
$$

yielding the σ-sextic $Q_3(x) = \varphi_1(x)\varphi_2(x)$, namely

$$
\begin{aligned}
Q_3(x) &= x^6 - x^5 + (-3\sigma^2 + 3\sigma - 5)\,x^4\\
&\quad +(2\sigma^2 - 2\sigma + 3)\,x^3\\
&\quad +(3\sigma^4 - 6\sigma^3 + 12\sigma^2 - 9\sigma + 7)\,x^2\\
&\quad -(\sigma^4 - 2\sigma^3 + 3\sigma^2 - 2\sigma + 1)\,x\\
&\quad -\sigma^6 + 3\sigma^5 - 7\sigma^4 + 9\sigma^3 - 9\sigma^2 + 5\sigma - 1.
\end{aligned}
$$

The cubics $\varphi_1(x)$, $\varphi_2(x)$ look very different from $\psi_1(x)$, $\psi_2(x)$ although both pairs represent the same physics, i.e. the same set of orbits. By eliminating σ between $S_3(\sigma)$ and either $\varphi_1(x)$ or $\varphi_2(x)$ we get back the original polynomial $H_3(x)$ of Eq. (1). Identical result is obtained eliminating r between $r^2 = 4a - 7$ and either $\psi_1(x)$ or $\psi_2(x)$. Comparing coefficients of equal powers in the sextics $H_3(x)$ and $Q_3(x)$ one recognizes that all coefficients are interconnected by the constraint $S_3(\sigma)$.

The discriminants of the $\varphi_1(x)$ and $\varphi_2(x)$ are

$$\Delta\varphi_1 = (4\sigma^2 - 6\sigma + 9)^2, \qquad \Delta\varphi_2 = (4\sigma^2 - 2\sigma + 7)^2.$$

Now, recall that "any third-degree polynomial $p(t) \in \mathbb{Q}(t)$ which is irreducible over the rationals $\mathbb{Q}$ will have a *cyclic* Galois group if and only if the discriminant of $p(t)$ is a square over $\mathbb{Q}$" [36]. Thus, $\Delta\varphi_1$ and $\Delta\varphi_2$ manifest clearly the advantage of σ-parameterization: It produces at once orbital equations with cyclic Galois group in a number-field of degree coinciding with the period of the orbits, i.e. the smallest number-field possible, yielding separated rather than entangled factors. This is not the case if we compute discriminants for $\psi_1(x)$ and $\psi_2(x)$. Although $\varphi_1(x)$ and $\varphi_2(x)$ are distinct functions, their discriminants with respect to σ are identical.

After this *excursus* emphasizing strength and generality of the method, let us consider what is encoded into $\varphi_1(x)$ and $\varphi_2(x)$. As the constraint $S_3(\sigma) = 0$ shows, two different values of σ lead to the same value of a. For instance, by taking either $\sigma = 0$ or $\sigma = 1$ we reach $a = 2$, the "partition generating limit" with many valuable properties [27], the limit where one may emulate logic gates, encode numbers, perform specific arithmetic operations on these numbers [32], and more [33]. For $\sigma = 0, 1$ we dispose of two independent microscopic σ-representations for each macroscopic state, namely

$$
\begin{aligned}
\Phi(x) &= \varphi_1(x;0) = \varphi_2(x;1) = x^3 - 3x - 1,\\
\overline{\Phi}(x) &= \varphi_1(x;1) = \varphi_2(x;0) = x^3 - x^2 - 2x + 1,
\end{aligned}
$$

where the overline is used to indicate that, in spite of their rather different functional forms, both functions are dynamically conjugated. These four expressions show that by permuting the values of σ we effectively *interchange* orbits, independently of the choice for $\varphi_\ell(x)$. Macroscopically in phase-space we deal with $\Phi(x)$ and $\overline{\Phi}(x)$. But microscopically the description may be done equally well using either $\varphi_1(x,\sigma)$ or $\varphi_2(x,\sigma)$. This degeneracy is not normally seen in phase-space [37].

Note that knowledge of just a single state, here $\varphi_1(x,\sigma)$ or $\varphi_2(x,\sigma)$, is enough to grant access to all physical states because the results obtained for one of them follow automatically for all *conjugate family* when we change the value of σ. For higher periods, conjugate families normaly contain hundreds of states. This is quite a lot because period-three implies chaos [34]. Thus, σ-the encoding stores conveniently all information concerning period-k dynamics for arbitrary k. It is a generic property of algebraic equations, not a peculiarity of the illustrative example considered. By suitably selecting σ one may switch from one orbit to another, performing a "delayed choice" of the ordering (labeling). By iterating polynomial automorphisms rather than orbital points one my even bypass the need for finding orbits in phase-space. In other words, orbits are automorphically correlated [37]. It is as if we were dealing with a multilevel "atom" in which the states could be defined and redefined by selecting the appropriate σ.

Do such parametric encodings exist also in more complicated multidimensional dynamical systems? Yes: the algebraic properties explored here are generic for dynamical systems of algebraic origin [26]. Incidentally, one-dimensional systems do not represent any restriction because multidimensional systems may be always reduced to one-dimensional equivalents [38]. For example, for the Hénon map $(x,y) \mapsto (a - x^2 + by, x)$, the prototypical multidimensional system which among other things describes very well the parameter space of class B lasers, CO_2 lasers in particular [39, 40], the generic cubic orbit encoding all period-3 solutions and valid for arbitrary values of the parameters a and b is

$$
\begin{aligned}
\mathcal{P}_3^H(x) &= x^3 - \sigma x^2 - \left[\sigma^2 - 2(1-b)\sigma + 3(1+b+b^2)\right]x\\
&\quad +\sigma^3 - 2(1-b)\sigma^2 + (3+b+3b^2)\sigma - 1 + b^3,
\end{aligned}
$$

where σ is now any root of the quadratic

$$\mathcal{S}_3^H(\sigma) = \sigma^2 - (1-b)\sigma + 2(1+b+b^2) - a.$$

In the fully dissipative $b = 0$ limit these equations correctly reproduce all results above. Parameterized equations covering the Hamiltonian limit ($b = -1$) and valid for all periods up to 22 are studied elsewhere [41]. Thus, one clearly sees that adding more parameters and/or extra dimensions only alter coefficients, not substance.

By adapting a concept developed for understanding quantum measurements we obtained a unified picture of what happens at the micro and macroscopic level of discrete-time classical dynamical systems. This new perspective is of course expected to apply equally well to more general situations, not only to algebraic systems. Although mathematical difficulties in deriving closed-form results for more intricate equations of motion greatly increase in this very general setup, no essential hindrances are anticipated to exist.

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