Absorptive optical bistability with laser-amplitude fluctuations

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Absorptive optical bistability with a Gaussian amplitude fluctuation of the injected signal is investigated. In the limit of large laser linewidth and good stabilization of the amplitude fluctuations, a Fokker-Planck equation for absorptive optical bistability is derived. The stationary probability distribution is derived and its properties are discussed.

I. INTRODUCTION

It has been recognized that in most realistic experiments dealing with a resonant atomic system driven by a laser the main source of noise is due to the partial coherence of the driving external laser light. The noise, i.e., the influence of possible laser fluctuations, can become very important if the driven system exhibits some critical behavior due to a nonlinear feedback mechanism. A very important example of such a system in quantum optics is one which exhibits the effect of optical bistability.[1] Optical bistability with laser fluctuations also provides a highly nontrivial example of an open system exhibiting a bifurcation and driven by an external source of noise. It is generally believed that in optical bistability laser phase and amplitude fluctuations of the injected laser signal can influence the bistable behavior of the system much more than the intrinsic quantum fluctuations of spontaneous emission.

The problem of quantum fluctuations in optical bistability has been investigated in detail by now.[2] The same is true for some external sources of noise leading to a white-noise multiplicative stochastic equation for optical bistability.[3,4] Except for a few numerical results[5,6] there is so far in the literature no detailed investigation of the influence of laser fluctuations on optical bistability. This is no doubt due to the difficulties associated with the nonwhite character of laser fluctuations in a nonlinear system far from equilibrium.

For quantum fluctuations a careful analysis of the quantum equations of motion for optical bistability has been performed. With the help of the Glauber P representation and after elimination of the atomic variables, a Fokker-Planck equation for the quasiprobability of the transmitted electric field can be obtained. In this case the noise, which is due to spontaneous emission, is basically white and the main complications come from the atomic degrees of freedom.

For an external noise due, for example, to laser fluctuations, the microscopic part of the dynamical system can be described by a set of simple macroscopic state equations which after a reduction of some of the degrees of freedom can be converted into a single nonlinear state equation, which shows explicitly the bistable behavior of the system. In this case, the dynamical part can be treated semiclassically and no microscopic complications enter the theory. The main complications which have prevented a closer examination of laser fluctuations come now from the source of noise itself.

A partially coherent laser has a finite bandwidth which results in a nonwhite noise in the nonlinear macroscopic state equation. A colored noise in this nonlinear dynamical state equation leads to very important problems. These problems are mostly related to properly deriving a generalized Fokker-Planck equation which correctly takes into account the colored character of the external laser noise.

In this paper we present an attempt toward the solution of this problem. We choose to discuss first a much simpler case of absorptive optical bistability (AOB) assuming an exact resonance of the laser beam with the optical resonator filled with a nondispersive absorber. We shall not take into account the phase fluctuations of the driving laser light. We know that phase fluctuations are perhaps much more important than amplitude fluctuations in realistic experiments,[7] but just in order to have some ideas of what an external nonwhite noise is doing to the hysteresis cycle we limit our discussion in this paper to Gaussian amplitude fluctuations of the injected signal.

The present paper is organized in the following way: In Sec. II we present a simple phenomenological stochastic equation of AOB in the limit of a good cavity, including Gaussian laser amplitude fluctuations. In Sec. III we derive for this nonlinear stochastic equation with an additive colored noise a Fokker-Planck equation in the limit of large laser linewidth and good amplitude stabilization.

This Fokker-Planck equation has a nonconstant diffusion term which shifts the most probable values of the steady-state probability distribution from its deterministic positions. In Sec. IV the stationary probability distribution of this Fokker-Planck equation is derived, and its properties, including the most probable position fluctuations and stability region, are discussed.

II. LANGEVIN EQUATION OF AOB

The theory of AOB in a cavity filled with homogeneously broadened two-level atoms is particularly simple in the limit of low-transmission mirrors and weak enough absorption. The critical properties of the optical bistability are given by the dimensionless order parameter $C=\alpha L/(2T)$, where α is the absorption coefficient of the medium, L is the length of the cavity, and T is the mirror transmission. In this limit (the mean-field limit) the relation between the transmitted x and the incident electric field y has the following form (in dimensionless units; see Ref. 1 for details):

$$\frac{dx}{dt}=-\left[x+\frac{2Cx}{1+x^2}\right]+y \;, \tag{2.1}$$

where the dimensionless time t is measured in units of the cavity bandwidth $\kappa\sim cT/(2L)$. In the steady-state limit ($\dot{x}=0$), Eq. (2.1) yields to the well-known state equation for real field amplitudes

$$y=x+\frac{2Cx}{1+x^2} \;, \tag{2.2}$$

which for $C>4$ exhibits a bistable behavior. In a realistic experimental situation the injected electric-field amplitude y should be replaced by a stochastic random field $y(t)$ describing the fluctuating electric-field envelope of the driving laser light.

From the laser theory[8] it is well known that for a laser operating far above threshold the total driving electric-field amplitude $y(t)$ consists of two components, a constant coherent part A_0 and a small fluctuation $\delta y(t)$, which is Gaussian with the following mean value and correlation function:

$$\langle \delta y(t)\rangle=0 \;, \quad \langle \delta y(t)\delta y(t')\rangle = a e^{-|t-t'|/b} \;, \tag{2.3}$$

where the dimensionless parameters a and b describe the characteristic properties of the fluctuating laser amplitude. The parameter a measures the intensity of the noise and $\sqrt{a}/y\sim 1/\sqrt{n_{ss}}$, where n_{ss} is the steady-state photon number of the laser operating far above threshold. The variance of amplitude fluctuations is equal to $\langle [y(t)]^2\rangle-\langle y(t)\rangle^2=a$ and in most realistic experiments $a\leq 0.1$, i.e., a better than 10% stabilization of the amplitude can be obtained.[7] The parameter $b=\kappa\tau_c$ is the coherence time τ_c of the laser light in units of the cavity linewidth. The inverse of τ_c gives the laser bandwidth Γ caused by amplitude fluctuations. For $b\to 0$ with $ab=D=$const, the stochastic process $\delta y(t)$, which is the well-known Ornstein-Uhlenbeck Brownian motion, tends to the white-noise limit (the Wiener-Levy stochastic process limit)

$$\langle \delta y_w\rangle=0 \;, \quad \langle \delta y_w(t)\delta y_w(t')\rangle=2D\delta(t-t') \;, \tag{2.4}$$

where $\delta y_w(t)=\lim_{b\to 0}\delta y(t)$ is a Gaussian process with a flat power spectrum and noise intensity equal to D. With amplitude fluctuations the AOB [Eq. (2.1)] takes the form of the following nonlinear Langevin equation with an additive Ornstein-Uhlenbeck stochastic process:

$$\frac{dx}{dt}=F(x)+\delta y \;, \tag{2.5}$$

where $F(x)$ is the deterministic part of the dynamical evolution given by

$$F(x)=-\left[x+\frac{2Cx}{1+x^2}\right]+A_0 \;. \tag{2.6}$$

This stochastic Langevin-type equation describes the AOB with Gaussian amplitude fluctuations. In the following sections we shall discuss the critical properties of this equation in the limit of weak fluctuations establishing a Fokker-Planck equation for the dynamical variable $x(t)$.

III. FOKKER-PLANCK EQUATION OF AOB WITH AMPLITUDE FLUCTUATIONS

In this section we shall derive a Fokker-Planck equation for the probability density of the transmitted field $x(t)$ governed by the stochastic equation (2.6). From this equation we derive first the following linear Liouville equation for the density $\varphi(t,x)=\delta[x-x(t)]$:

$$\frac{\partial}{\partial t}\varphi(t,x)=[M_0+i\delta y(t)M]\varphi(t,x) \tag{3.1a}$$

with the linear differential operators

$$M_0=-\frac{\partial}{\partial x}F(x) \quad \text{and} \quad M=i\frac{\partial}{\partial x} \;. \tag{3.1b}$$

The Fokker-Planck probability distribution function $P(t,x)$ is obtained from the density $\varphi(t,x)$ by performing a stochastic average over all possible realizations of the random noise $\delta y(t)$

$$P(t,x)=\langle \varphi(t,x)\rangle \;. \tag{3.2}$$

We can write a formal but exact expression for $P(t,x)$ by solving Eq. (3.1) and using the definition (3.2) to obtain

$$P(t,x)=e^{M_0 t}V(t)P(0,x) \;, \tag{3.3}$$

where

$$V(t)=\left\langle T\exp\left[i\int_0^t ds\,\delta y(s)M_I(s)\right]\right\rangle \tag{3.4a}$$

is the "evolution" operator of Eq. (3.1) averaged over $\delta y(t)$. The operator $M_I(s)=e^{-M_0 s}M\,e^{M_0 s}$ corresponds in form to the interaction picture of quantum mechanics and T denotes the standard time-ordering operator. For a Gaussian stochastic process $\delta y(t)$ an exact expression for $V(T)$ can be obtained[9,10]

$$V(t)=T\exp\left[-\tfrac{1}{2}\int_0^t ds_1\int_0^t ds_2 M_I(s_1)\times\langle \delta y(s_1)\delta y(s_2)\rangle M_I(s_2)\right] \;. \tag{3.4b}$$

Owing to the noncommutativity of $M_I(t)$ at different times this exact but formal solution (3.4b) does not lead to mathematically tractable expressions for $P(t,x)$. The noncommutativity leads to an infinite hierarchy of equations for $P(t,x)$. The structure and the form of this hierarchy can be investigated using standard methods of quantum-field theory applied to statistical mechanics of open systems. Different mathematical techniques ranging from chronologically ordered cumulants,[9] the Baker-Campbell-Hausdorf formula,[11] path integration,[10] or van Kampen's diagrams,[12] have been used in the discussion of this problem. For the purpose of this paper we shall use a very simple argument in order to obtain the lowest nontrivial and nonperturbative contribution to $V(t)$.

We have assumed that the intensity of the noise amplitude $\delta y(t)$ is proportional to $a<1$ [see Eq. (2.3)]. From the structure of the solution (3.4) we see that by incorporating $\sqrt{a}$ into M_I the commutator $[M_I(t),M_I(t')]$ becomes proportional to a and, accordingly, higher-order commutators of $M_I(t)$ at different times are proportional to higher powers of a. Because $a<1$ we shall assume that in the lowest-order approximation in a $[M_I(t),M_I(t')]\cong 0$, i.e., the chronological product in Eq. (3.4) is redundant. This permits us to calculate from Eqs. (3.3) and (3.4) the following closed-form equation for $P(t,x)$:

$$\frac{\partial}{\partial t}P(t,x)=\left[M_0-\int_0^t ds\, Me^{-L(t-s)}\langle \delta y(t)\delta y(s)\rangle M\right]\times P(t,x)\ , \tag{3.5}$$

where the Liouville operator L acts on M as follows: $LM=[M,M_0]$. Equation (3.5) is a Fokker-Planck equation of AOB with small ($a<1$) amplitude fluctuations. More sophisticated arguments based on different mathematical techniques (e.g., time-ordered cumulants) lead to the same Fokker-Planck equation in the limit of small fluctuations.[12] This Fokker-Planck equation is still rather complicated due to the complex form of its right-hand side (rhs), involving the Liouville operator. To exhibit this we perform a formal integration of the time-dependent part of the rhs operator in (3.5) with the correlation function (2.3)

$$\int_0^t ds\, e^{-L(t-s)}\langle \delta y(t)\delta y(s)\rangle = a\frac{1-e^{-(L+1/b)t}}{L+1/b}\ . \tag{3.6}$$

If $b<1$, i.e., when the laser linewidth is larger than the cavity bandwidth, i.e., with $\Gamma>\kappa$, we can expand the inverse of the Liouville operator in Eq. (3.6) as a Neuman series

$$\frac{1}{L+1/b}=b(1-bL+\cdots)\ . \tag{3.7}$$

For $b<1$ we shall keep only the first terms in the power series (3.7). For $t>b$ we obtain from Eqs. (3.6) and (3.7) the following Fokker-Planck equation for the AOB with Gaussian amplitude fluctuations:

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial x}FP+D\frac{\partial^2}{\partial x^2}KP\ , \tag{3.8}$$

where the nonconstant diffusion function $K(x)$ has the form:

$$K(x)=(1+bF')=1-b-2bC\frac{1-x^2}{(1+x^2)^2}\ , \tag{3.9}$$

and where $D=ab$.

We check that in the limit of a white noise given by Eq. (2.4), i.e., if $b\to 0$, then $K\to 1$ and the diffusion term takes the well-known constant form. In this case only the constant parameter D plays the role of the diffusion. In general, for $b\neq 0$, the diffusion function (3.9) depends on the laser linewidth b. It is possible to calculate a proper diffusion function in Eq. (3.5) for arbitrary values of b, summing up the entire Neuman series (3.7).[13] However, for the purpose of this work and in order to obtain analytic results, we limit ourselves to the case of $b<1$, i.e., to the diffusion function given by Eq. (3.9).

In order to describe a proper Fokker-Planck equation, the diffusion function given by Eq. (3.9) must be positive: $K(x)\geq 0$. This condition is fulfilled for all values of x only if

$$b\leq\frac{1}{1+2C}\ . \tag{3.10}$$

This condition is consistent with the power-series expansion (3.7) which is valid only for $b<1$. Note also that the distribution $P(t,x)$ derived in this section is not associated to any particular ordering prescription of the photon creation and annihilation operators, e.g., normal or antinormal, because quantum fluctuations are neglected.

IV. STEADY-STATE PROBABILITY DISTRIBUTION

We write the stationary solution $[(\partial/\partial t)P_{st}=0]$ of the Fokker-Planck equation (3.5) in the following form:

$$P_{st}=N\,e^{-U(x)/D}\ , \tag{4.1}$$

where N is a normalization constant and $U(x)$ is a generalized "nonequilibrium potential" which may be compared with an equilibrium potential like the free energy. From the Fokker-Planck equation (3.5) we derive the following exact thermodynamical potential assuming natural boundary conditions:

$$U(x)=-\int dx\left[\frac{F(x)}{1+bF'(x)}\right]+ab\ln|1+bF'(x)|\ . \tag{4.2}$$

In this expression F' denotes a derivative of the function F given by Eq. (2.6) with respect to its argument.

States of maximal probability for AOB with laser amplitude fluctuations correspond to the absolute minimum of $U(x)$. Setting $dU/dx=0$ we obtain the following "state equation":

$$F(x)-ab^2F''(x)=0\ . \tag{4.3}$$

We see that the condition (4.3), which gives the most probable values of the stationary probability distribution, depends on the product ab^2 of the two coefficients a and b which characterize the laser amplitude fluctuations. For a flat power spectrum, i.e., for $b\to 0$, the potential

$$U\underset{b\to 0}{\to}-\int^x dx'F'(x') \tag{4.4}$$

and the most probable values of x correspond to the deter-

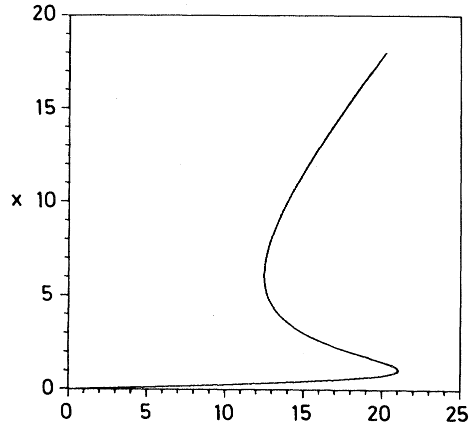

FIG. 1. Location of the extrema of the stationary probability for $C=20$, $a=0.4$, and $b=0.023\,17$.

ministic state equation $F(x)=0$ [see Eq. (2.2) and Ref. 2]. The impact of a finite bandwidth of the injected laser mode on the critical properties of the AOB is clear from Eq. (4.3). This equation leads to the following relation for the most probable values of the stationary probability distribution $p_{st}(x)$:

$$A_0=\left[x+\frac{2Cx}{1+x^2}\right]+4ab^2C\frac{3-x^2}{(1+x^2)^3}\ . \tag{4.5}$$

Equation (4.5) can be regarded as a generalization of the deterministic relation (2.2) for the case of laser amplitude fluctuations. We believe expression (4.5) to be the first explicit formula for the AOB with realistic laser amplitude fluctuations. Note that in absence of noise the state equation leads to the deterministic curve $y=A_0(x)$ given by Eq. (2.2). This deterministic curve has an inflection point, the "critical point" with horizontal tangent. In the presence of the noise the critical point depends also on the parameter ab^2 which characterizes the amplitude fluctuations. Furthermore, note that in the limit of white-noise fluctuations [see Eq. (2.4)] the state equation (4.5) reduces to the deterministic relation (2.2). Only a nonwhite (i.e., $b\neq 0$) character of the additive external noise has a nontrivial influence on the thermodynamical properties of the bistable system.

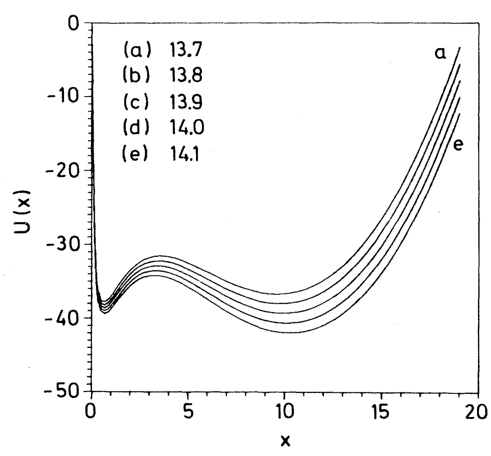

FIG. 2. These curves show the shape of the thermodynamical potential $U(x)$ ($C=20$, $a=0.4$, and $b=0.023\,17$) for values of the input A_0 field as indicated.

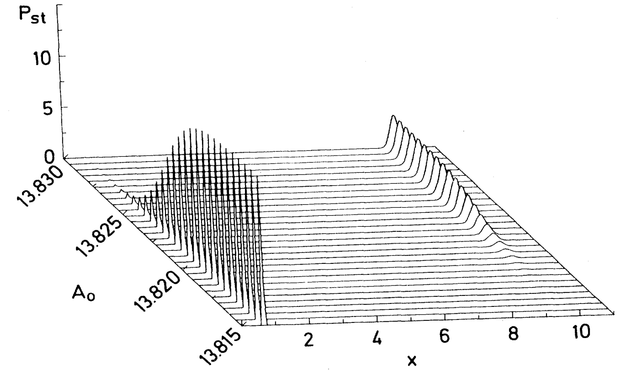

FIG. 3. Development of the stationary probability ($C=20$, $a=0.4$, and $b=0.023\,17$) as a function of the injected laser amplitude A_0. System exchanges global stability at $A_0=13.824\,75$.

We have investigated the critical properties of AOB for the order parameter $C=20$ and with laser fluctuations characterized by $a=0.4$ and $b=0.023\,17$, fixed for all further calculations.[14] These external noise parameters correspond to a noisy laser with linewidth b^{-1} times larger than the cavity bandwidth [see Eqs. (2.3) and (3.10)]. For these values of laser fluctuations the parameter ab^2 in Eq. (4.5) is very small ($ab^2=2.148\times 10^{-4}$) and no significant change of the most probable values of the stationary probability distribution given by Eq. (4.5) with respect to the deterministic case can be observed. Figure 1 shows the perfect overlap of Eq. (4.5) with the deterministic bistable hysteresis. However, before drawing final conclusions from the state equation (4.5) we have to realize that it describes only the most probable values of the stationary probability distribution (4.1). To make a real comparison of the theory with included laser fluctuations with the deterministic case we have to discuss the relative stability of the bistable branches. This can be achieved by comparing the relative depth of the minima of $U(x)$. In this way we can predict the point of equal probability, i.e., the point where the two branches exchange global stability. An explicit integration in Eq. (4.2) can be performed leading to an analytical formula for the nonequilibrium potential $U(x)$. The analytical expression is long and has been included as the Appendix.

In Fig. 2 we have plotted $U(x)$ for $a=0.4$ and $b=0.023\,17$ and different values of the input field A_0. It is clear from the explicit form of the potential $U(x)$ given in the Appendix that the depth and the width of the bistable minima depend on the laser parameters in a much more complicated way than the factor ab^2 appearing in the condition (4.5) for the most probable values of $P_{st}(x)$. As in the case of quantum fluctuations[15,2] the random amplitude of the laser field leads to a small range of values of A_0 in which the two peaks have a comparable area. In Fig. 3 we have plotted the stationary probability for different values of the coherent input laser field A_0. Figure 4 shows the mean value and the normalized fluctuation $(\langle x^2\rangle-\langle x\rangle^2)/\langle x\rangle^2$ for $C=20$, $a=0.4$, and $b=0.023\,17$. Clearly, the mean value of the transmitted field $\langle x\rangle$ coincides with one of the two deterministic branches except in the narrow transition region where we have large fluctuations.

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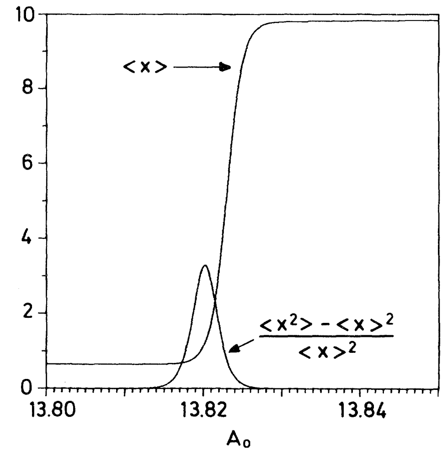

FIG. 4. Mean value $\langle x\rangle$ and the fluctuations $(\langle x^2\rangle-\langle x\rangle^2)/\langle x\rangle^2$ of the transmitted field vs the input laser amplitude A_0 for $C=20$, $a=0.4$, and $b=0.023\,17$.

APPENDIX

The integration in Eq. (4.2) can be performed analytically. As a result we obtain the following formula for the thermodynamic potential:

$$U(x)=-\frac{A_0}{1-b}\left[x+\frac{\beta(1+f)}{h\sqrt{f}}\arctan\frac{x}{\sqrt{f}}-\frac{\beta(1+g)}{h\sqrt{g}}\arctan\frac{x}{\sqrt{g}}\right]+\frac{1}{2(1-b)}\left[x^2+\left[C-\frac{\beta}{2}\right]\ln|(1+x^2)^2+\beta(x^2-1)|\right]$$
$$+\frac{\beta(\beta+4-2C)}{2h}\ln\left|\frac{x^2+f}{x^2+g}\right|+ab\ln\left|1-b-2Cb\frac{1-x^2}{(1+x^2)^2}\right| ,$$

where

$$\beta=\frac{2Cb}{1-b},\quad h^2=\beta(8+\beta),\quad g=1+\tfrac{1}{2}(\beta+h),\quad f=1+\tfrac{1}{2}(\beta-h) .$$

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