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Recovering parameters from self-similar structures in phase space of dissipative systems with constant Jacobian

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Abstract

We show that dissipative dynamical systems with constant Jacobian allow one to recover numerical values of control parameters under which the system is operating. This is done by performing measurements on self-similar (fractal) structures in phase space. We illustrate parameter recovery explicitly for the Ikeda laser ring-cavity map and the Hénon map. The first model involves transcendental equations of motion and can be solved only numerically. Analytical results are obtained for the Hénon map. In both cases, the dissipation rate is recovered from the speed at which fractal "fingers" making up basins of attraction accumulate towards basin boundaries.

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This note reports equations defining interesting and generic *paths* (*surfaces*) in the parameter space of dissipative systems with constant Jacobian. The main characteristic of such paths is that under a suitable condition (existence of self-similar structures in phase space) they allow one to retrieve numerical values of model parameters under which the system is operating. Such paths are simple consequences of a standard linear

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stability analysis around fixed points [1] and are obtained by relating two independent quantities easily derived from the Jacobian matrix of the system: the eigenvalue of largest magnitude and the determinant J of the Jacobian matrix, i.e. the Jacobian of the map. Since parameter paths depend on the largest eigenvalue we call them *eigenvalue paths*. Hence, this note shows how to derive eigenvalue paths from the Jacobian of dissipative systems.

We illustrate parameter recovery for the Ikeda laser ring-cavity map [2–4] and the Hénon map [5]. The first model involves transcendental functions and parameters must be recovered numerically. Results for the Ikeda model are of great interest because they may be checked experimentally [2–4]. Wide-ranging analytical results are easy to obtain for the Hénon map. The possibility of recovering parameters from the geometrical structuring of the phase space is a very fruitful theoretical twist providing rather privileged parameter paths for performing a plethora of numerical experiments: to investigate metamorphoses of basin boundaries, 'explosions' of chaotic attractors, to identify recurring arithmetical properties underlying bifurcation cascades, etc.

We start with the simpler case, deriving the eigenvalue path for the Hénon map $(x,y) \mapsto (a - x^2 + by, x)$. The dissipation rate is given by the Jacobian $J = -b$. The volume contraction for dissipative systems can be evaluated using the Lie derivative along the dynamical vector fields. The eigenvalues of the Hénon map are $\lambda_\pm = -x_u \pm (x_u^2 + b)^{1/2}$, where (x_u, x_u) is the location of the unstable fixed point: $2x_u = b - 1 - \{(b-1)^2 + 4a\}^{1/2}$. Then, from the relation $\lambda_+ = 1/J = 1/(-b)$ connecting contraction and dissipation, after some simple algebra we find the eigenvalue path

$$W(a,b) \equiv 4ab^2 - (b^2 + b + 1)(b^2 + 3b + 1)(b-1)^2 = 0. \tag{1}$$

Of particular interest along this path is the point [6,7] $p=(a,b)=(-9b^*/2,\ b^*)$, where $b^* = -2 + \sqrt{3} \simeq -0.267949192$. Three different curves meet at p: the eigenvalue path, the $1 \to 2$ bifurcation boundary and the period-3 saddle-node bifurcation line. As may be seen from Fig. 4 of Ref. [6], the phase space at p has a self-similar structure and is subdivided into three basins: B_∞, of the trivial attractor at infinity, B_3, of the period-3 attractor, and B_1, of the period-1 attractor. There is also an unstable fixed point $u = (x_u, x_u)$, $x_u = -(9 - 3\sqrt{3})/2 \simeq -1.901923$, on the boundary of B_∞. We use an auxiliary horizontal line passing through u and intersecting the infinite sequence of self-similar vertical doublets of stripes which compose B_3 (see Fig. 4 in Ref. [6]). Each doublet constitutes a "finger" $f = 1, 2, \ldots$ (in the direction towards u in B_∞) and is characterized by four coordinates $x_i^{(f)}$ delimiting the intervals containing the stripes. Thus, the interval (finger) $[x_4^{(f)}, x_1^{(f)}]$ contains two stripes: $[x_4^{(f)}, x_3^{(f)}]$ and $[x_2^{(f)}, x_1^{(f)}]$.

The 13 fingers given in Table 1 are the first of an infinite quantity accumulating towards u. The regularity of the accumulation process motivates us to measure two characteristic speeds: (i) v_a, the "accumulation speed", i.e., the rate at which fingers accumulate towards B_∞, and (ii) v_c, the "compression speed", i.e., the rate with which the pair of stripes composing the fingers get "compressed" as they move closer and

Table 1

f	$x_1^{(f)}$	$x_2^{(f)}$	$x_3^{(f)}$	$x_4^{(f)}$
1	−1.1752108557	−1.1982723739	−1.4189347616	−1.4353771639
2	−1.6943339156	−1.7016203204	−1.7661597167	−1.7711038446
3	−1.8452938554	−1.8473190561	−1.8651178820	−1.8664743531
4	−1.8866774615	−1.8872252421	−1.8920312312	−1.8923969125
5	−1.8978333562	−1.8979805007	−1.8992709032	−1.8993690458
6	−1.9008273880	−1.9008668416	−1.9012127935	−1.9012391021
7	−1.9016299822	−1.9016405557	−1.9017332668	−1.9017403170
8	−1.9018450615	−1.9018478948	−1.9018727377	−1.9018746268
9	−1.9019026936	−1.9019034528	−1.9019101095	−1.9019106157
10	−1.9019181362	−1.9019183397	−1.9019201233	−1.9019202590
11	−1.9019222741	−1.9019223286	−1.9019228065	−1.9019228429
12	−1.9019233828	−1.9019233974	−1.9019235255	−1.9019235352
13	−1.9019236799	−1.9019236838	−1.9019237181	−1.9019237207

closer to B_∞. Thus, from the coordinates $x_i^{(f)}$ we obtain the ‘instantaneous speeds’

$$v_a^{(f)} = \frac{x_1^{(f+2)} - x_1^{(f+1)}}{x_1^{(f+1)} - x_1^{(f)}}, \qquad v_c^{(f)} = \frac{x_1^{(f)} - x_2^{(f)}}{x_3^{(f)} - x_4^{(f)}}, \tag{2}$$

and the asymptotic limits $v_a = \lim_{f\to\infty} v_a^{(f)}$, and $v_c = \lim_{f\to\infty} v_c^{(f)}$. The data in Table 1 above yields two *exponential* laws

$$v_a^{(f)} = v_a + \exp[-1.33\ f - 2.40], \qquad v_c^{(f)} = v_c + \exp[-1.32\ f - 1.03],$$

where $v_a \simeq 0.267949195$ and $v_c \simeq 1.49978549$. In both cases, the magnitude of the characteristic exponent is close to $\frac{4}{3}$. Comparing v_a with $J = -b = 2 - \sqrt{3} = 0.267949192$ we find that $v_a - J = 3 \times 10^{-9}$, an impressive agreement. Similar agreement was obtained for other reference lines not tangent to the basin boundary or otherwise too particularly placed and for more than 30 additional parameters on the eigenvalue path.

The laser ring-cavity map [2] is $z_{t+1} = \alpha\, e^{i\theta} z_t + \beta$, where $\theta = \Delta - \delta/(1 + |z_t|^2)$ and, as usual, $\Delta = 0.4$ and $\delta = 6$. The Jacobian is α^2. For $0.786 < \alpha < 0.86766$, $(0.70 < \beta < 1.22)$, the eigenvalue path of the laser ring-cavity map is well approximated by $\beta = 5.24182 - 4.00875\alpha - 1.41208\alpha^2$. A particularly interesting point on this path is $p^* = (\alpha, \beta) = (0.84753, 0.83)$ where one finds not only a period-3 structure of fingers as before but also an intricate *finger-within-finger* substructuring. This additional substructure is the basin of a period-36 orbit, a surprisingly high period, multiple of the period of the basin ‘containing’ it. The unstable fixed point u is located near $(x, y) = (1.7509, -2.1924)$. At p^* we have $\alpha^2 = 0.71830$ while the dissipation measured from the period-3 fingers is 0.71828. The same dissipation rate is obtained from the period-36 fingers. Similar agreement is obtained for many other parameter values on the eigenvalue path. Eigenvalue paths may contain parameter subintervals not containing fingers (no self-similar structures in phase space), when parameter recovery becomes obviously impossible. The dissipation, however, remains always connected to the eigenvalue.

In conclusion, we have shown explicitly that physical parameters of dissipative dynamical systems with constant Jacobian may be recovered from measurements done solely on the geometrical self-similar structuring of their phase space. A detailed account of the structure of the eigenvalue paths will be presented elsewhere.

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