Physics Letters A 375 (2010) 143–148

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

How similar is the performance of the cubic and the piecewise-linear circuits of Chua?

Gonzalo M. Ramírez-Ávila [a,b], Jason A.C. Gallas [b,c,d,*]

[a] Institut für Physik, AG Nichtlineare Dynamik, Humboldt Universität zu Berlin, 12489 Berlin, Germany
[b] Instituto de Investigaciones Físicas, Universidad Mayor de San Andrés, Cota Cota, La Paz, Bolivia
[c] Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brazil
[d] Departamento de Física, Universidade Federal da Paraiba, 58051-970 João Pessoa, Brazil

ARTICLE INFO

Article history:
Received 16 September 2010
Received in revised form 12 October 2010
Accepted 22 October 2010
Available online 26 October 2010
Communicated by A.R. Bishop

Keywords:
Nonlinear dynamics
Phase diagram
Parameter isomorphism
Complex systems
Periodicity hub

ABSTRACT

This Letter reports phase diagrams quantifying and contrasting the dynamical performance of the paradigmatic piecewise-linear and cubic circuits of Chua. Although both circuits may be regarded as macroscopically isomorphic over wide regions in control parameter space, we show that their microscopic structure displays a myriad of rather distinctive intrinsic features making them unique. Inhomogeneities embedded in periodic and chaotic phases complicate some applications of the circuits but may also adequately act as realistic noise proxies in synchronization problems. In addition, infinite cascades of spirals and hubs observed experimentally very recently in a related dissipative flow are shown to be also present in both circuits of Chua, emerging however in a rather distinctive asymmetric way. Thus Chua's circuits may be used to study experimentally elusive and theoretically intricate phenomena generating periodicity hubs.

1. Introduction

The paradigmatic circuit of Chua has been continuously at the forefront of research during the last 25 years as a fruitful testground for theoretical and experimental advancements in nonlinear dynamics which now fill several books [1–5] and many articles, e.g. [6–14]. Quite recently, this circuit has allowed the experimental observation of novel complex structures and phenomena in parameter space, like the so-called "shrimps" [15–18] which were observed both isolated [19,20] or, in a slightly different setup, forming infinite spirals connected to certain remarkable "periodicity hubs" [21,22,14,23]. The popularity of Chua's circuit is enhanced by the great reliability of electronic circuits and the excellent agreement normally found between measured and predicted behaviors. In fact, this characteristic of circuits allows one to probe novel devices and theories with very high accuracy [21–24].

According to a widespread opinion held about Chua's circuit (Fig. 1), both piecewise-linear and cubic circuits display dynamical behaviors which are "similar" [1–5]. Such similarities are usually elicited by comparing relevant dynamical quantities for a few selected parameters or, sometimes, by comparing bifurcation diagrams along specific sections of the control parameter space. However, some applications, e.g. synchronization of networks of identical and non-identical circuits, require a much more specific assessment of the degree of similarity of its constituents. Thus, synchronization of a large number of Chua's circuit leads one to ask the question whose answer is the focal point of this work: how quantitatively similar is the dynamical performance of circuits with piecewise-linear and cubic nonlinearities? In other words, how complete is the operational isomorphism of this pair of circuits? Could eventual differences in behavior act as realistic proxies for the ubiquitous noise seen in real-world systems? These are the question that we address here.

To quantify how similar both circuits behave we computed numerically high-resolution phase diagrams over extended parameter ranges (see Figs. 2–5 below). Synthetically, the general conclusion is that although on a coarse-grained scale both nonlinearities may be regarded as *macroscopically* isomorphic over wide regions in control space independently of the parameters tuned, their *microscopic* structure displays rather distinctive features.

For instance, while Chua's circuit contains periodicity hubs similar to the ones reported recently in lasers and other systems [14, 23–27], hubs in Chua's circuit present asymmetries and peculiarities which distinguishes them from everything seen so far (see below). A plethora of microscopic inhomogeneities between periodic and chaotic phases poses a number of challenges to efficient applications.

* Corresponding author at: Instituto de Física, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brazil.
E-mail address: jgallas@if.ufrgs.br (J.A.C. Gallas).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.10.046

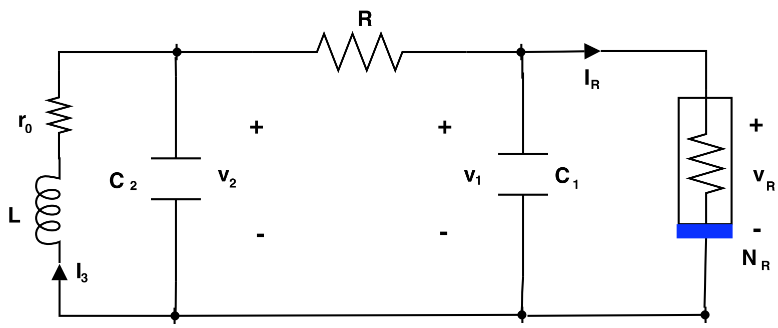

Fig. 1. The basic circuit leading to Eqs. (1a)–(1c), containing three control parameters $\alpha = C_2/C_1$, $\beta = R^2C_2/L$ and γ, defined as a function of the four linear elements C_1, C_2, L, and R in which the third parameter depends on the linear resistor r_0 in series with the inductor ($\gamma = Rr_0C_2/L$).

The aim of this work is to contrast the control parameter space of Chua's circuits with piecewise-linear and cubic nonlinearities. The main messages here are (i) that the widespread "equivalence" of both circuits is in fact not valid and their dynamical behavior needs to be carefully asserted for each specific application, and (ii) that the phase diagrams of both circuits display a rather rich structure, with many features which are not yet understood theoretically. Before starting, let us mention that there has been a number of recent investigations concerning the parameter space of both flavors of Chua's circuit, cubic and piecewise-linear [7–14], in particular concerning the identification of period-adding cascades in them [11,13].

2. The nonlinear circuits

As schematically shown in Fig. 1, Chua's circuit contains five linear elements (two capacitors, one inductor, and two resistors) and a nonlinear element, the so-called Chua's diode (N_R), playing the role of a negative resistor, and which normally contains two additional parameters [1–5]. In dimensionless form, the circuit is governed by the equations [6]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha\bigl(y - x - f(x)\bigr), \tag{1a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y + z, \tag{1b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\beta y - \gamma z, \tag{1c}$$

where $f(x)$ stands for the nonlinearity and, in terms of the basic reactances, the three basic control parameters are

$$\alpha = \frac{C_2}{C_1}, \qquad \beta = \frac{R^2C_2}{L}, \qquad \gamma = \frac{Rr_0C_2}{L}. \tag{2}$$

Originally [28], the function $f(x)$ was taken as

$$f(x) = bx + \frac{1}{2}(a-b)\bigl(|x+1| - \bigl(|x-1|\bigr)\bigr), \tag{3}$$

where a and b are free parameters controlling the diode N_R. But a popular variant involves replacing this piecewise-linear function by diode with a smooth cubic characteristic [6]

$$f(x) = \hat{a}x^3 + \hat{b}x, \tag{4}$$

where $\hat{a}$ and $\hat{b}$ are free parameters. This cubic preserves the odd-symmetric character of the original piecewise function. The dynamical behavior of the piecewise-linear variant has been studied extensively and found not to capture correctly all features of a real circuit [29]. The main relevance of the cubic nonlinearity is that nonlinear devices are always smooth in real circuits [6]. Several works dealt with a smooth nonlinearity in Chua's oscillator [30–32]. The cubic nonlinearity has been implemented experimentally [29] and widely studied. For a comprehensive survey see Tsuneda [6].

To compare the performance of both circuits one first needs to ensure that they operate as identically as possible. To this end, we maximize the identity of the nonlinearities $f(x)$ above by suitably selecting the parameters $\hat{a}$ and $\hat{b}$ of the cubic to match a given pair (a, b). This is done by a least-square fit of the cubic to the piecewise linear function over an interval of approximation $[-d, d]$, a procedure which produces a parameter "bridge" among both circuits [6,33,34]:

$$\hat{a} = -35\bigl(d^2-1\bigr)^2(a-b)/\bigl(16d^7\bigr), \tag{5a}$$

$$\hat{b} = b + \bigl(45d^4 - 50d^2 + 21\bigr)(a-b)/\bigl(16d^5\bigr). \tag{5b}$$

Following previous workers [2], we fix $a = -8/7 \simeq -1.1428$, $b = -5/7 \simeq -0.7142$, and $d = 2$ a choice that gives $\hat{a} = 0.0659$ and $\hat{b} = -1.1671$.

3. Phase diagrams

We start now by illustrating how changes in the reactances affect solutions and stability of both circuits by computing and comparing Lyapunov phase diagrams [14] for them. Of particular interest is to compare changes in shape and volume of both periodic and chaotic phases and, more importantly, the details of their inner structure.

Fig. 2 shows phase diagrams illustrating relevant portions of the $\alpha \times \gamma$ parameter plane for each circuit, a plane containing particularly rich mixture of periodicity and chaotic phases. As indicated by the color scales, gray shadings signal parameter regions characterized by periodic solutions (negative Lyapunov exponents), while colors always mark chaotic phases (positive exponents). The bluish coloration in Fig. 2(a) represents the piecewise linear circuit while the greenish hue is used for the cubic nonlinearity. Each panel in our figures displays $1200 \times 1200 = 1.44 \times 10^6$ Lyapunov exponents. The large pink domains indicate parameters for which most initial conditions lead to unbounded solutions. Fig. 2 was computed for $\beta = 1000$ and presents "asymptotic" phase diagrams, in the sense that they essentially remain invariant when β is further increased. The basic structure in both panels of Fig. 2 show that while the overall coarse-grained distribution of chaos and periodicity in Fig. 2 looks similar at first sight, their precise distribution has a large number of small differences that are hard to summarize efficiently with words.

Fig. 3 illustrates parameter isomorphism when tuning β from 10 to 100, i.e. when moving from the low β region to the asymptotic limits shown in Fig. 2. Again, while there is an overall agreement of the dynamics observed for both circuits when increasing β, the phase diagrams are not identical. In lose words, while it

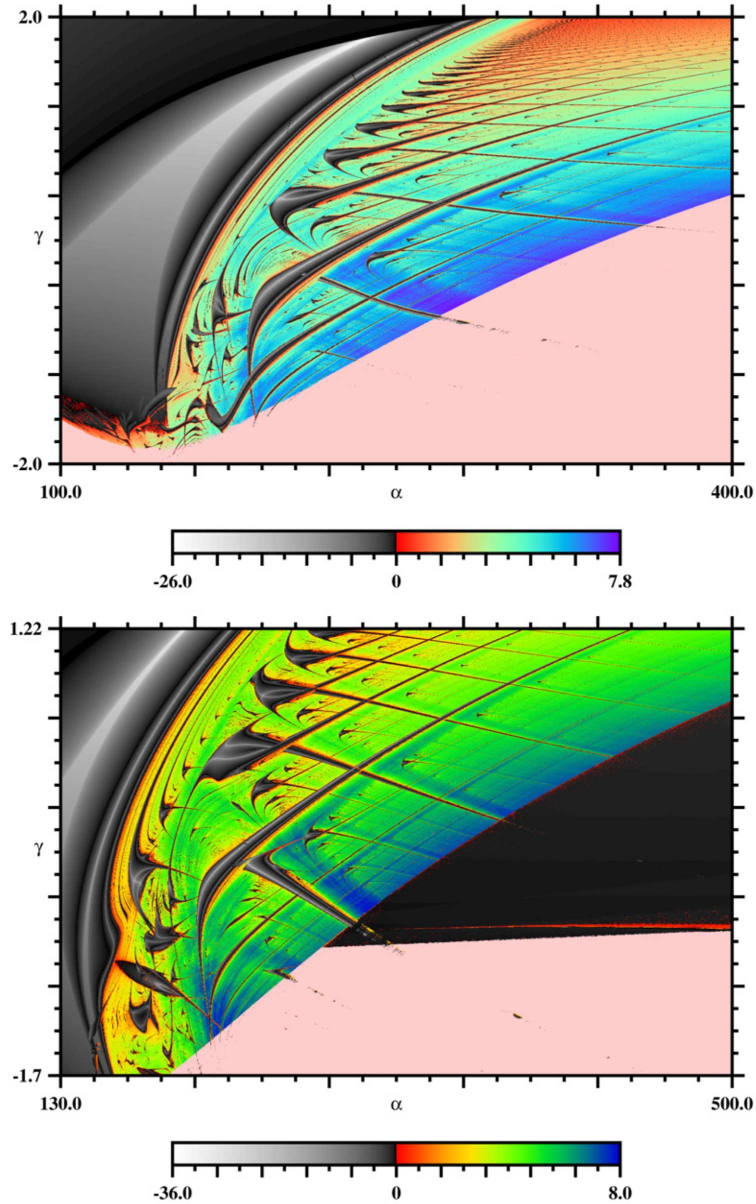

Fig. 2. (Color online.) Parameter space isomorphism between the piecewise linear (top) and cubic (bottom) circuits. The global structure is quite similar and the magnitude of the Lyapunov exponents is almost identical. Here $\beta = 1000$.

is possible to argue the existence of a "macroscopic" isomorphism, microscopically the phase space structure is different. The impact of such differences depends on the application intended for the circuits. For instance, arrays of coupled circuits have their asymptotic dynamical behavior strongly influence by the precise structure of the individual oscillators. In this sense, although complicating some applications of the circuits, inhomogeneities embedded in the periodic and/or chaotic phases also act quite conveniently as noise proxies in the stabilization of the synchronization properties of large networks of oscillators [35].

Fig. 4 compares three parameter regions where one sees particularly intricate behaviors. The leftmost panels illustrate the presence of spirals and periodicity hubs which are typically observed for low values of the parameters, in the present case for $\beta = 10$. Such hubs were recently found to organize the stable dynamics into a regular alternation of periodic and chaotic phases over large portions of parameter space [7,8,14,23]. Although the global views of parameter space presented in Figs. 2 and 3 were selected to enhance similarities, there is obviously no exact correspondence among the parameter domains shown. In contrast, in Fig. 4 we compare the structure of identical parameter windows for both circuits. While the complicated alternation of stable chaos and periodicity seen in the middle panels of Fig. 4 defies any attempt of describing them with words, requiring pictures to describe the situation, the regular organization present in the rightmost panel is considerably more tame. In fact, at first sight the phase diagram in Fig. 4(c) might seem relatively similar to the one in Fig. 4(f). However, that this is not the case is illustrated in Fig. 5 which shows a rather different distribution of dark islands of regular oscillations embedded in the colored chaotic phases.

In connection with the hubs illustrated in Figs. 4(a) and 4(d), we point out that *incomplete* homoclinic scenarios were recently measured ground-breaking experimental studies by Al-Naimee et al. [36,37] in a semiconductor laser with optoelectronic feedback. Subsequently, such laser system was found [26] to contain cascades of spirals of stable oscillations and hubs which look identical to the familiar ones observed when in presence of *complete* homoclinic scenarios. This all means that hubs like the ones in

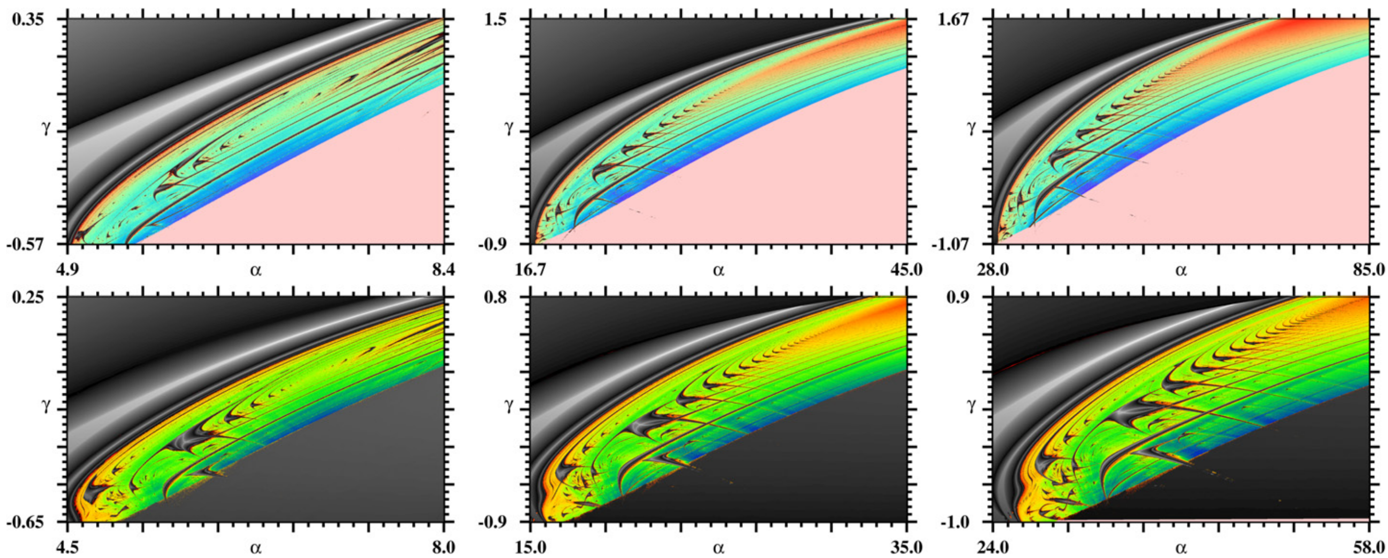

Fig. 3. (Color online.) Low β isomorphism between phase diagrams for the piecewise-linear (top row) and cubic (bottom row) circuits for $\beta = 10, 50, 100$, from left to right. Although stretched over extended parameter regions, as β varies the overall trend of the circuits remains essentially the same. The leftmost column displays periodicity hubs, shown magnified in Fig. 4. Each panel displays 1200×1200 Lyapunov exponents.

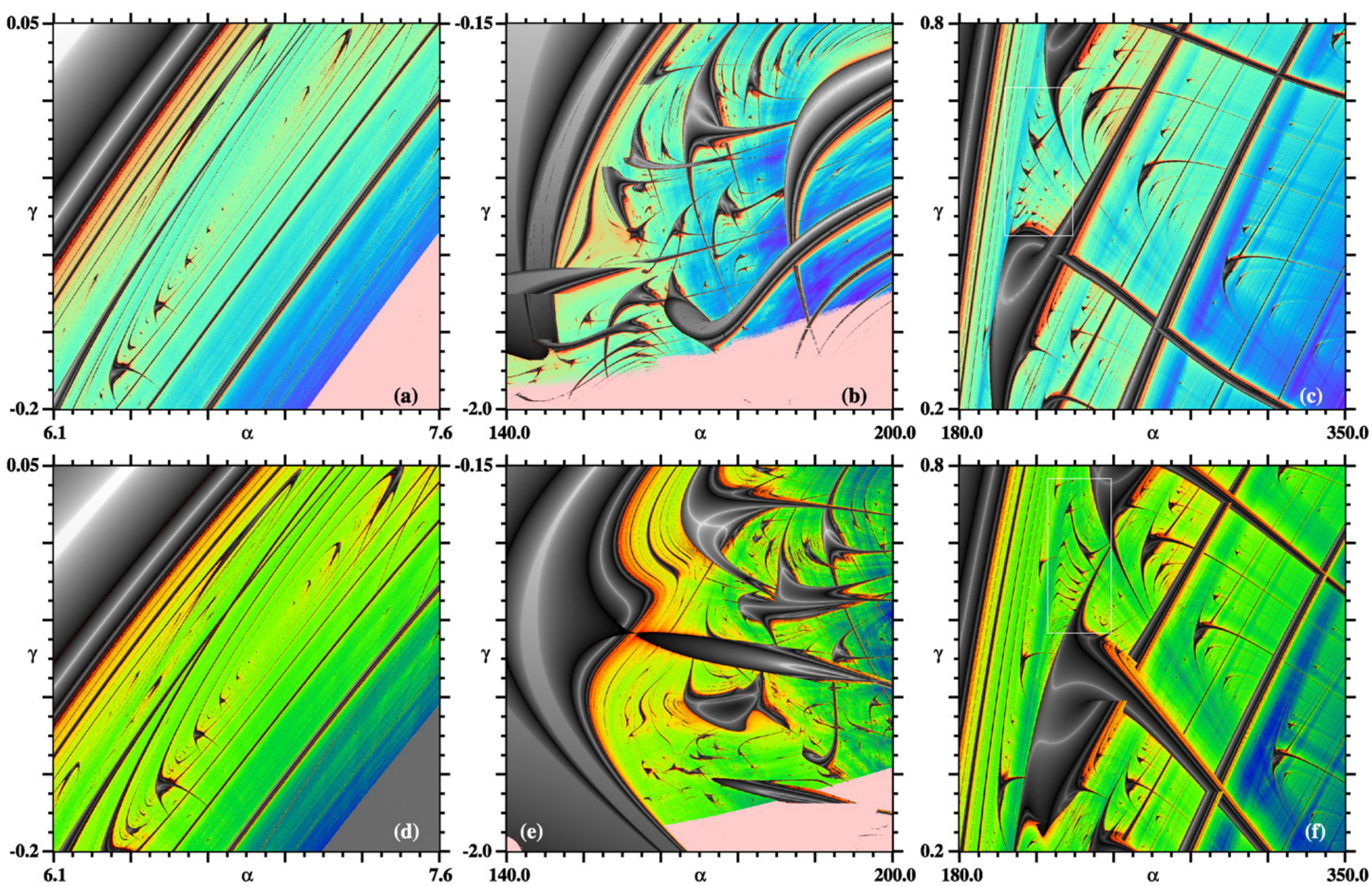

Fig. 4. (Color online.) Finer details of the phase diagrams for the piecewise-linear circuit (top row) and the cubic circuit (bottom row). The leftmost panels show periodicity hubs [7,8,14,23] for $\beta = 10$ while the other panels are for $\beta = 1000$. Note the abundance of multistability, particularly easy to recognize in the middle column panels. The white boxes in the rightmost panels are shown magnified in Fig. 5.

Figs. 4(a) and 4(d) could be far more general then theoretically presumed so far, being not necessarily limited by the dynamics usually attributed to Shilnikov's theorem. This fact opens the possibility of using Chua's circuit to measure complex distributions of oscillations and addressing their nature, Shilnikov or non-Shilnikov.

4. Conclusions and outlook

In summary, comparison of the parameter space topology between the piecewise-linear and cubic circuits reveals a number of similarities, an isomorphism, only when regarded on a relatively coarse scale. As exemplified by Fig. 5, the microscopic structure

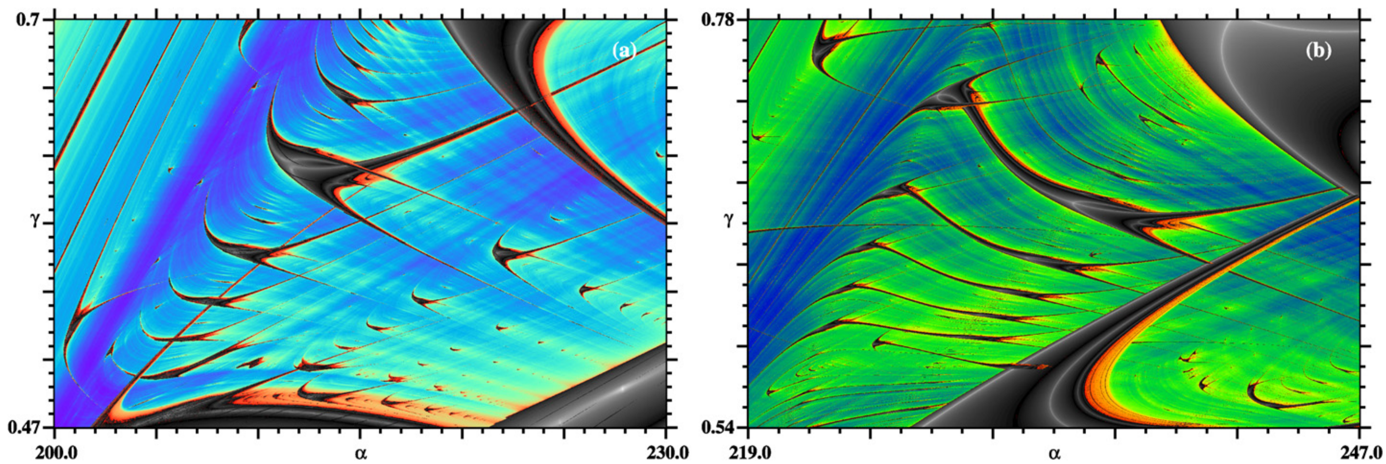

Fig. 5. (Color online.) Enlargements of the white boxes in Figs. 4(c) and 4(f) illustrating significant differences of the periodic phases. The cubic circuit displays a cascade of "xiphopagus shrimps" not present in the piecewise-linear circuit. Differences in colors reflect the fact that the color scale of each panel is not fixed but renormalized according the local minimum and maximum Lyapunov exponents. Each panel displays 2400×2400 exponents.

of both circuits displays a myriad of quite distinctive intrinsic features which make them individually rather unique. Since measurements with Chua's circuit are not difficult to carry out [1–3], it would be interesting to check how faithfully experimental phase diagrams reproduce the distribution of periodic and chaotic phases reported here. Of particular interest is to check whether or not the cascade of "conjoined twin shrimps" in Fig. 5(b) could be attributed to slight differences among the nonlinearities of both circuits.

One important result discussed here is the unequivocal presence of periodicity hubs and spirals in both the piecewise-linear and the cubic circuits. This means that by suitably tuning parameters along spirals, each one characterized by oscillations with specific waveforms, one may navigate towards hubs, focal accumulation points. At such remarkable focal point in parameter space it is possible to commute from an incoming to an outgoing spiral *in an infinite number of ways*, each spiral corresponding to a characteristic family of stable oscillations, periodic or not [see Fig. 4(a) and (d)]. Incidentally, the study of hubs and spirals sheds additional light on the parameter space structure related with period-adding cascades associated with them, e.g. in semiconductor lasers subjected to optical injection [38,25,26], chemical reactions [39], and other dissipative flows, piecewise-linear or not [23,14]. For recent developments about the complexities connected with period-adding cascades in Chua's circuit see the interesting contributions by Albuquerque and coworkers [11,13].

Theoretically, there is an enticing novel twist connected with our present investigation that deserves attention. In certain situations, hubs are known to be directly linked to Shilnikov's homoclinic scenario [14,40–48]. However, the possibility of non-Shilnikov scenarios has been illustrated recently for a semiconductor laser with optoelectronic feedback [26]. At present, it is yet totally unclear whether or not hubs exist in more general setups and how to anticipate their existence. This fact was a strong motivation for our work: to reveal promising scenarios to probe the dynamics observed around hubs. There is no theory to predict the presence of periodicity hubs and only numerical or experimental work is presently capable of detecting them. We remark that while it is very tempting to associate periodicity hubs with homoclinic orbits and a theorem by Shilnikov, numerical work shows hubs and spirals not to exist in typical flows that are textbook examples of the Shilnikov setup [23,26]. This means that under Shilnikov conditions, hubs can either exist or not. The discovery of hubs in Chua's circuit is thus an enticing novel twist that may be now used to study the elusive and theoretically complex phenomena responsible for their genesis. As illustrated by our phase diagrams, the parameter space of both circuits contains a plethora of intricate features whose detailed origin and global unfolding is a big challenge that remains to be accounted for theoretically.

Acknowledgements

GMRA is supported by the German Academic Exchange Service (DAAD). JACG is supported by CNPq, Brazil, and by the Air Force Office of Scientific Research, grant FA9550-07-1-0102. All computations were done in the computer clusters of the CESUP-UFRGS.

References

[1] R. Kiliç, A Practical Guide for Studying Chua's Circuit, World Scientific, Singapore, 2010.
[2] L. Fortuna, M.G. Xibilia, Chua's Circuit Implementations, World Scientific, Singapore, 2009.
[3] E. Bilotta, P. Pantano, A Gallery of Chua Attractors, World Scientific, Singapore, 2008.
[4] M. Ogorzalek, Chaos and Complexity in Nonlinear Electronic Circuits, World Scientific, Singapore, 1997.
[5] R. Madan, Chua's Circuit: A Paradigm for Chaos, World Scientific, Singapore, 1993.
[6] A. Tsuneda, Int. J. Bifur. Chaos 15 (2005) 1.
[7] G.M. Ramirez-Avila, J.A.C. Gallas, Rev. Boliviana Fis. 14 (2008) 1.
[8] G.M. Ramirez-Avila, J.A.C. Gallas, Rev. Boliviana Fis. 18 (2011), in press.
[9] H.A. Albuquerque, R.M. Rubinger, P.C. Rech, Phys. Lett. A 372 (2008) 4793.
[10] H.A. Albuquerque, P.C. Rech, Int. J. Bifur. Chaos 19 (2009) 1351.
[11] H.A. Albuquerque, P.C. Rech, http://dx.doi.org/10.1002/cta.713, Int. J. Circ. Theor. Appl., in press.
[12] E.R. Viana, R.M. Rubinger, H.A. Albuquerque, A.G. de Oliveira, G.M. Ribeiro, Chaos 20 (2010) 023110.
[13] C. Stegemann, H.A. Albuquerque, P.C. Rech, Chaos 20 (2010) 023103.
[14] For a survey see J.A.C. Gallas, Int. J. Bifur. Chaos 20 (2010) 197, and references therein.
[15] E.N. Lorenz, Physica D 237 (2008) 1689.
[16] J.A.C. Gallas, Phys. Rev. Lett. 70 (1993) 2714.
[17] J.A.C. Gallas, Physica A 202 (1994) 196.
[18] J.A.C. Gallas, Appl. Phys. B 60 (1995) S203.
[19] M.S. Baptista, M.B. Reyes, J.C. Sartorelli, C. Grebogi, E. Rosa, Int. J. Bifur. Chaos 13 (2003) 2551.
[20] D.M. Maranhão, M.S. Baptista, J.C. Sartorelli, I.L. Caldas, Phys. Rev. E 77 (2008) 037202.
[21] R. Stoop, P. Benner, Y. Uwate, Phys. Rev. Lett. 105 (2010) 074102.
[22] R. Stoop, P. Benner, Y. Uwate, in: R. Tetzlaff, W. Schwarz, K. Kelber (Eds.), Proceedings of the 18th IEEE Workshop on Nonlinear Dynamics of Electronic Systems, NDES, Dresden, 2010, pp. 6–9.
[23] C. Bonatto, J.A.C. Gallas, Phys. Rev. Lett. 101 (2008) 054101.
[24] C. Bonatto, J.C. Garreau, J.A.C. Gallas, Phys. Rev. Lett. 95 (2005) 143905.
[25] V. Kovanis, A. Gavrielides, J.A.C. Gallas, Eur. Phys. J. D 58 (2010) 181.

[26] J.G. Freire, J.A.C. Gallas, Phys. Rev. E 82 (2010) 037202.
[27] M.A. Nascimento, J.A.C. Gallas, H. Varela, Phys. Chem. Chem. Phys. 12 (2010), in press.
[28] T. Matsumoto, S. Tanaka, L.O. Chua, Phys. Rev. A 30 (1984) 1155.
[29] G.Q. Zhong, IEEE Trans. Circuits Syst. I 41 (1994) 934.
[30] A.I. Khibnik, D. Roose, L.O. Chua, Int. J. Bifur. Chaos 3 (1993) 363.
[31] G.V. Osipov, V.D. Shalfeev, IEEE Trans. Circuits Syst. I 42 (1995) 687.
[32] Q. Yuan, X.S. Yang, Int. J. Bifur. Chaos 18 (2008) 2391.
[33] E.J. Altman, J. Circuits Syst. Comput. 3 (1993) 63.
[34] E.J. Altman, IEEE Trans. Circuits Syst. II 40 (1993) 675.
[35] A. Uchida, F. Rogister, J. Garcia-Ojalvo, R. Roy, in: E. Wolf (Ed.), in: Progress in Optics, vol. 48, Elsevier, Dordrecht, 2005, pp. 203–341.
[36] K. Al-Naimee, F. Marino, M. Ciszak, R. Meucci, F.T. Arecchi, New J. Phys. 11 (2009) 073022.
[37] K. Al-Naimee, F. Marino, M. Ciszak, S.F. Abdalah, R. Meucci, F.T. Arecchi, Eur. Phys. J. D 58 (2010) 187.
[38] C. Bonatto, J.A.C. Gallas, Phys. Rev. E 75 (2007) 055204(R);
C. Bonatto, J.A.C. Gallas, Phil. Trans. Royal Soc. London 366A (2008) 505.
[39] J.G. Freire, K. Showalter, J.A.C. Gallas, Phase diagrams for a three-variable autocatalator model of mixed-mode oscillations, preprint (2010).
[40] P. Glendinning, Stability, Instability and Chaos, Cambridge University Press, Cambridge, 1994.
[41] L.A. Belyakov, Math. Notes 15 (1974) 336.
[42] L.A. Belyakov, Math. Notes 28 (1981) 910.
[43] L.A. Belyakov, Math. Notes 36 (1985) 838.
[44] P. Glendinning, C. Sparrow, J. Stat. Phys. 35 (1984) 645.
[45] P. Gaspard, R. Kapral, G. Nicolis, J. Stat. Phys. 35 (1984) 697.
[46] F.T. Arecchi, Chaos 1 (1991) 357.
[47] T. Braun, J.A. Lisboa, J.A.C. Gallas, Phys. Rev. Lett. 68 (1992) 2770.
[48] R. Vitolo, P. Glendinning, J.A.C. Gallas, preprint (2010).