Vertices in Parameter Space: Double Crises Which Destroy Chaotic Attractors

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We report a new phenomenon observed along a crisis locus when two control parameters of physical models are varied simultaneously: the existence of one or several *vertices*. The occurrence of a vertex (loss of differentiability) on a crisis locus implies the existence of *simultaneous* sudden changes in the structure of both the chaotic attractor and of its basin boundary. Vertices correspond to *degenerate* tangencies between manifolds of the unstable periodic orbits accessible from the basin of the chaotic attractor. Physically, small parameter perturbations (noise) about such vertices induce drastic changes in the dynamics.

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In the theory of dissipative systems one often studies phenomena such as bifurcation of periodic orbits, routes to chaos, etc., as a single parameter is varied. Although changing single parameters is common in practice, physical models usually contain several *tuning* or *control* parameters that might be varied to exhibit a characteristic of interest or to avoid undesirable behaviors. The question then arises as to what interesting dynamics exists if more than one parameter is changed simultaneously. New phenomena are likely to occur when adjusting more than one parameter at the same time. We therefore ask what is to be expected in situations where it is possible or desirable to change more than one parameter simultaneously. Of particular interest to nonlinear systems are *crisis loci* in parameter space, i.e., the locus of those parameters where boundary crises [1] occur. This locus corresponds to parameter values which separate regions characterized by drastically different kinds of dynamical behavior. More specifically, recall [1] that as a single parameter (say p) varies, the distance between an existing chaotic attractor and its basin boundary decreases until, at some critical value $p=p_c$, the attractor touches its basin boundary, characterizing the boundary crisis phenomenon. Beyond p_c the chaotic attractor (and its basin of attraction) no longer exists but is replaced by a chaotic transient after which the system settles down on another attractor, frequently not chaotic. If one then varies the parameter in the opposite direction, a chaotic transient is converted to a chaotic attractor at $p=p_c$. We mention that the phenomenon of crisis is a generic route to chaos which has been studied as a single parameter is varied. This route has often been observed experimentally; cf. Ref. [2] for some references. While tuning a single parameter, Refs. [1] and [2] considered changes happening while *going across* the crisis locus at p_c. The purpose of this Letter is to report a new phenomenon observed while *moving along* the crisis locus. To determine a crisis locus one has to vary two parameters *simultaneously*: one is needed to assure the existence of the crisis and another is used to see how the crisis evolves in the parameter space. By definition, the chaotic attractor for parameters chosen along the crisis locus keeps touching its basin boundary. Therefore, a relevant question is then what happens to the basin boundary when the chaotic attractor undergoes an *interior crisis* [1] with a sudden, say, increase in size. Or, conversely, what happens to the chaotic attractor when the basin undergoes a *metamorphosis* [3] with a sudden increase in its extent? We find that along the locus of crisis, both interior crisis and metamorphoses have to occur simultaneously. In other words, when the chaotic attractor increases in size, the boundary shrinks, and when the boundary increases its extent, the attractor shrinks. In addition, since the attractor is also touching the boundary, a boundary crisis is occurring simultaneously. Therefore, the phenomenon we are describing consists of the simultaneous occurrence of a *double* crisis plus a metamorphosis. Its characteristic signature is the appearance of a *vertex* on the crisis locus, a point in which the locus loses its differentiability. Because of the occurrence of a metamorphosis at the vertex, there is also a change in the unstable periodic orbit accessible from the basin of the chaotic attractor. All that means is that the dynamics is extremely rich at such vertices.

To illustrate a particular crisis locus, let us now consider a model of a laser ring cavity. In quantum optics, the behavior of a laser ring cavity is described [4–6] by the equation

$$z_{t+1}=\alpha e^{i\theta}z_t+\beta\,, \tag{1}$$

known as the Ikeda map. In this equation, the complex variable $z_t=x_t+iy_t$ represents the electric field at the beginning of the tth passage around the ring, α is the coefficient of reflectivity of the partially reflecting output mirror, while β is related to the laser input amplitude. The quantity θ is a relatively complicated functional of the laser field inside the cavity. Ikeda [4] considered $\theta=|z_t|^2$ while Hammel, Jones, and Moloney [5] used

$$\theta=\Delta-\frac{\delta}{1+|z_t|^2}\,, \tag{2}$$

Δ being the empty cavity detuning and δ giving an additional detuning when a nonlinear medium is present. We

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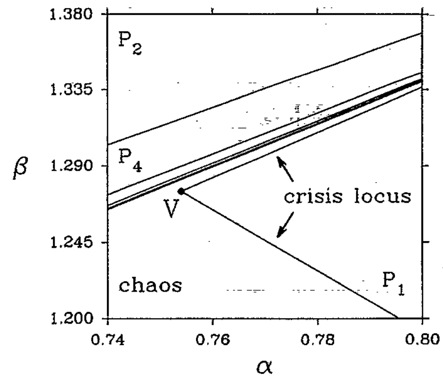

FIG. 1. Crisis locus for the laser ring cavity map, displaying a sharp discontinuity at the vertex V. P_i refers to the asymptotic periodicity of the region. Both segments are nearly straight lines, sometimes having tiny gaps [9].

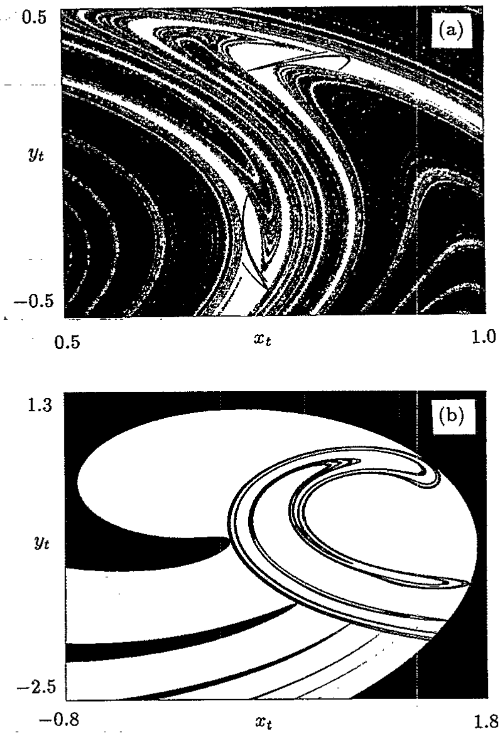

FIG. 2. Differences between typical chaotic attractors and their basins as seen for the laser ring cavity map with parameters along different segments of the crisis locus of Fig. 1: (a) above V, $\beta=1.317$ and (b) below V, $\beta=1.222673$; for both cases $\alpha=0.784$. Note the differences in scales.

will consider the latter model and, without any loss of generality, study the dynamics obtained by varying the reflectivity α and the laser amplitude β for fixed detunings $\delta=6$ and $\Delta=0.4$

Figure 1 shows regions in the parameter space characterized by displaying a common asymptotic attractor when Eq. (1) is iterated for a ball of initial conditions around, say, $(x_0,y_0)=(0,0)$. The subindex i of the P_i inside each region denotes the observed periodicity of the corresponding asymptotic periodic attractors. Moving from top to bottom in the diagram one sees period-doubling cascades that accumulate on the thicker line, below which there is a region of parameters for which one predominantly finds chaotic behavior. The roughly triangular shape on the right delimits a region inside which the asymptotic behavior is a stable fixed point. The letter V marks a vertex on the crisis locus while the two lines converging to it correspond to portions of the crisis locus. This locus is the borderline between parameters corresponding to two quite different attractors: chaotic, to the left of the locus, and periodic, to the right. As aforementioned, however, the dynamics observed *along* the crisis locus is the main subject of this paper. Chaotic attractors along the upper line converging to V consist of two disconnected pieces while attractors along the lower line are made of one single piece. Figure 2 shows two typical examples of such attractors and their basins, both for $\alpha=0.784$. White regions represent the basin of the chaotic attractors while black regions correspond to initial conditions leading to periodic attractors. Figure 2(a) is obtained for β lying on the upper branch of the crisis locus in Fig. 1. Along that branch the attractor consists of two pieces as seen in Fig. 2(a), while on the lower branch the attractor is connected as shown in Fig. 2(b). The essential difference between Figs. 2(a) and 2(b) is that in Fig. 2(a) the attractor is smaller than in Fig. 2(b), being in fact just a portion of the bigger attractor. Inside the wedge the chaotic attractor does not exist anymore, with its former basin becoming now part of the basin of another attractor (of period 1) located far from the ball of original initial conditions. The basin corresponding to the two-piece attractor along the line converging to V is fractal. At crisis there is typically a saddle periodic orbit that is simultaneously in the basin boundary and in the chaotic attractor. This "crisis orbit" has period 6 along the upper branch in Fig. 1, while the crisis orbit of the lower branch has period 1. The vertex V, located roughly at $(\alpha,\beta)\simeq(0.754,1.275)$, corresponds to the point where the two segments meet.

To illustrate further the generality of the above phenomenon, we consider now the Hénon map [7,8] which is a paradigm in the study of nonlinear systems. It is defined by the equations

$$x_{t+1}=a-x_t^2+by_t,\quad y_{t+1}=x_t\,. \qquad (3)$$

Figure 3 shows the crisis locus for Eq. (3). The crisis locus, made up of the simple curves [9] connected at the vertices V_i, was obtained by following the attractor while slowly increasing a from $a=0$ along lines of constant b. The "unbounded" region in Fig. 3 shows parameters for which almost all initial conditions lead to divergence. The "bounded" region corresponds to predominantly chaotic attractors [8]. As in the case of the ring cavity,

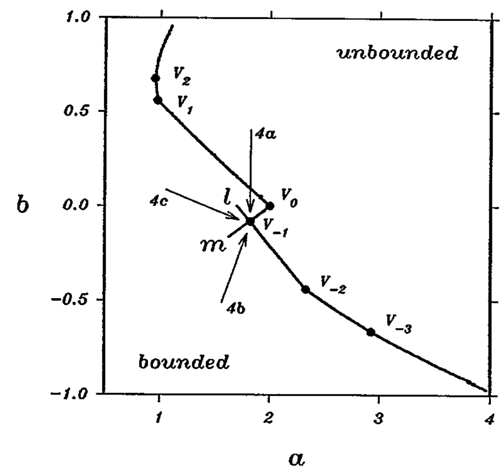

FIG. 3. Crisis locus for the Hénon map showing several vertices. Salient features are listed in Table I. The phase space of points 4*a*, 4*b*, and 4*c* (chosen very close to V_{-1}) are given in Figs. 4(a), 4(b), and 4(c).

the Hénon crisis locus also contains vertices where the dynamics changes abruptly. The vertex V_0 at $(a=2, b=0)$ marks the crisis for the one-dimensional quadratic map. Notice that the vertices V_i are subscripted with negative values when $b<0$. Table I collects several important characteristics of the vertices: their location in parameter space, the type of basin boundary for crisis points near the vertices, the changes in the number of pieces of the chaotic attractors as one goes across the vertices, and, finally, the changes in the periods of the crisis orbits that occur at the vertices. We point out that the segments composing the crisis locus that meet a vertex continue (in some sense) to exist past the vertex but as points where interior crises [1] occur. For example, as seen in Fig. 3, the segments l and m extend to the left of the vertex V_{-1} into the region of chaos as a locus of interior crisis and metamorphosis, respectively.

To understand the dynamics at the vertices we magnify Fig. 3 in the neighborhood of the vertex V_{-1} and plot in Fig. 4 the chaotic attractors and part of their basins of attraction in three of the quadrants delimited by the lines l and m. In the fourth or right quadrant, almost all initial conditions lead to orbits diverging to infinity. Figure 4(a)

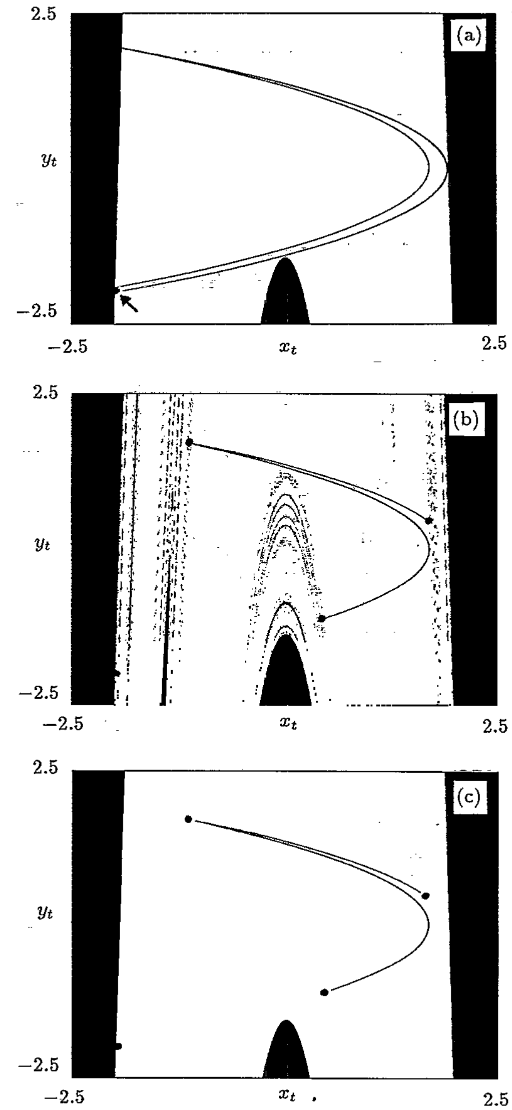

FIG. 4. Basins of attraction and corresponding chaotic attractors for the Hénon map in the neighborhood of V_{-1}: (a) upper quadrant with arrow pointing the accessible period-1 orbit for $a=1.817$ and $b=-0.080$; (b) lower quadrant showing the accessible period-3 orbit for $a=1.817$ and $b=-0.084$. The fractal structure precludes accessibility of the period-1 orbit from the white region; (c) left quadrant showing the accessible period-1 and period-3 orbits for $a=1.797$ and $b=-0.082$.

TABLE I. Parameter values and salient changes at the vertices of crisis, as b decreases. F→NF, for example, means a transition from "fractal to nonfractal."

Vertex	a	b	Basin boundary	No. of pieces of attractors	Period of the crisis orbit
V_2	0.9488525	0.677	F→F	4→2	12→6
V_1	0.973005	0.55861	F→NF	2→1	6→1
V_0	2.0	0.0	NF→NF	1→1	1→1
V_{-1}	1.817	−0.082	NF→F	1→1	1→3
V_{-2}	2.327915	−0.44129	F→F	1→2	3→6
V_{-3}	2.923444	−0.663939	F→F	2→4	6→12

corresponds to parameters in the upper quadrant slightly above V_{-1}. The basin boundary is a one-dimensional nonfractal curve and is the closure of the stable manifold of an accessible [10] unstable period-1 orbit while the attractor is a one-piece attractor which lies in the closure of the unstable manifold of the same accessible orbit. Figure 4(b) corresponds to parameters in the lower quadrant. The basin boundary is in this case fractal and is the closure of the stable manifold of a period-3 unstable orbit while the attractor is in the closure of the unstable manifold of the same orbit. The relevant question here is then how the system makes the transition from the situation depicted in Fig. 4(a) to that in Fig. 4(b) in which the boundary has increased in its extent while the attractor was forced to shrink accordingly. The answer lies in Fig. 4(c), the link between Figs. 4(a) and 4(b). Figure 4(c) corresponds to parameter values in the left quadrant and shows the situation as seen slightly before the tangency of the chaotic attractor and the period-3 orbit. In this figure, the basin boundary is the closure of the accessible period-1 orbit while the attractor lies in the closure of one of the branches of the unstable manifold of the unstable period-3 orbit lying in the basin of attraction [the *same* period-3 orbit shown in Fig. 4(b)]. The transition from Fig. 4(a) to 4(c), crossing the l line in Fig. 3, results in an interior crisis with a discontinuous jump in size of the chaotic attractor. The transition from Fig. 4(c) to Fig. 4(b), crossing the m line in Fig. 3, results in the smooth-to-fractal metamorphosis [3] of the basin boundary with a discontinuous change in its extent and in its dimension. Similar transitions occur at all other vertices, with the exception of V_0. This is because the $b=0$ line is special in that the system is no longer two-dimensional but collapses instead to a one-dimensional map. With the sole purpose of simplifying the discussion, the present Letter was restricted to situations involving just *two* parameters. For more than two parameters, vertices can be correspondingly more complex. Then, they are the intersections of three planes in the crisis locus.

In conclusion, by analyzing the parameter space of two familiar and representative models, we described the dynamics along crisis loci and showed them to have one or several *vertices*. As seen in Fig. 4, small fluctuations (noise) in parameters around such vertices will induce sizable changes in the dynamics and in the final observed physical behavior. We argued that dynamically vertices correspond to degenerate tangencies between manifolds of accessible periodic orbits and signal places where exchange of periodicity of the unstable accessible orbits at the basin boundary occur. The phenomenon creating vertices and their occurrence is generic and, in fact, is found to be quite typical in the parameter space of physical models when more than one parameter is varied simultaneously.

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[1] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **48**, 1507 (1982); Physica (Amsterdam) **7D**, 181 (1983).

[2] J. Elezgaray and A. Arneodo (to be published); D. Dangoisse, P. Glorieux, and D. Hennequin, Phys. Rev. Lett. **55**, 746 (1985); E. G. Gwinn and R. M. Westerfeld, Phys. Rev. Lett. **54**, 1613 (1985); R. W. Rollins and E. R. Hunt, Phys. Rev. A **29**, 3327 (1984); H. Ikezi and J. S. de Grasse, Phys. Rev. A **28**, 1201 (1983); S. K. Bronson, D. Dewey, and P. S. Linsay, Phys. Rev. A **28**, 1201 (1983); C. Jeffries and J. Perez, Phys. Rev. A **27**, 601 (1983); W. L. Ditto *et al.*, Phys. Rev. Lett. **63**, 923 (1989); J. C. Sommerer *et al.*, Phys. Lett. A **153**, 105 (1991).

[3] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **56**, 1011 (1986); Physica (Amsterdam) **24D**, 243 (1987).

[4] K. Ikeda, Opt. Commun. **30**, 257 (1979); K. Ikeda, H. Daido, and O. Akimoto, Phys. Rev. Lett. **45**, 709 (1980).

[5] S. Hammel, C. K. R. T. Jones, and J. Moloney, J. Opt. Soc. Am. **2**, 552 (1983).

[6] C. O. Weiss and R. Vilaseca, *Dynamics of Lasers* (VCH Verlagsgesellschaft, Weinheim, 1991).

[7] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).

[8] J. A. C. Gallas, Phys. Rev. Lett. **70**, 2714 (1993); Int. J. Mod. Phys. C **3**, 1295 (1992); HLRZ Report No. 48/93 [Physica (Amsterdam) A (to be published)].

[9] While most crisis loci are expected to be continuous curves, in some small intervals they might appear as *piecewise* continuous if the initial conditions are not properly "tuned." These regions contain cascades of period doubling $k2^n$. Color pictures of such intervals along the path of crisis of the Hénon map can be seen in Ref. [8].

[10] In this Letter, "accessible orbit" always refers to the unstable periodic orbit on the boundary which is accessible from the basin of the chaotic attractor; see Ref. [3].